REACTORS WITH STELLARATOR STABILITY AND TOKAMAK TRANSPORT

Paul R. Garabedian[1], Long-Poe Ku[2] and the ARIES Team

[1]*Courant Institute of Mathematical Sciences, New York University, New York, NY 10012* garabedian@cims.nyu.edu
[2]*Princeton Plasma Physics Laboratory, Princeton University, Princeton, NJ 08543* lpku@pppl.gov

The discovery of quasiaxially symmetric stellarators whose magnetic spectrum has approximate two-dimensional symmetry opens up the possibility of designing fusion reactors that have tokamak transport and stellarator stability. Prototypes with two or three field periods have asymmetries almost as small as the coefficients for a typical tokamak that are associated with ripple from the toroidal coils or helical excursion of the magnetic axis resulting from instability. We have found modular coils that are only moderately twisted and produce robust flux surfaces that do not deteriorate when changes are made in the magnetic field. This work is bolstered by recent stellarator experiments that have exceeded stability limits predicted by linear theory. The problem may be that force balance and stability are lost across islands if the equilibrium equations are not in conservation form.

I. INTRODUCTION

Quasiaxially symmetric stellarators (QAS) are a class of optimized stellarators for which the spectrum of the magnetic field **B** has approximate two-dimensional symmetry. The discovery of such stellarators brings the three-dimensional toroidal confinement concept of plasmas closer to the tokamak. This is true for the National Compact Stellarator Experiment (NCSX) at the Princton Plasma Physics Laboratory (PPPL) [1], which has three field periods, plasma aspect ratio A = 4.5, and rotational transform ι in the range $0.66 > \iota > 0.35$, and for the Modular Helias-like Heliac 2 (MHH2) [2], which is a configuration with two field periods, plasma aspect ratio 2.5, and rotational transform in the range $0.55 > \iota > 0.40$. The asymmetric terms in a typical QAS spectrum are almost as small as the coefficients for a comparable tokamak that are associated with ripple from the toroidal coils or helical modes resulting from magnetohydrodynamic (MHD) instability [3].

The physics of toroidally confined plasmas calls for a mathematical description by fully three-dimensional models. Mathematical analysis shows that differentiable solutions of the problem of three-dimensional toroidal equilibrium in plasma physics do not exist. This has resulted in problems with convergence of some of the computer codes used to design and analyze stellarators. An appropriate response to that difficulty is to apply an equilibrium and stability code like NSTAB that employs a conservation form of the magnetostatic equations to calculate weak solutions that capture discontinuities modeling effectively both current sheets and small chains of magnetic islands. The TRAN Monte Carlo transport code has been coupled with NSTAB to give physically realistic estimates of the energy confinement time in stellarators and tokamaks that are based on three-dimensional bifurcated equilibria.

II. EQUILIBRIUM AND STABILITY

The NSTAB equilibrium and stability code applies the MHD variational principle

$$\iiint \left[\frac{1}{2}B^2 - p(s)\right]dV = \text{minimum} \qquad (1)$$

to compute weak solutions of a conservation form of the partial differential equations of magnetostatics [2]. The solutions are represented in terms of Clebsch potentials by the formulas

$$\mathbf{B} = \nabla s \times \nabla\theta = \nabla\phi - \zeta\nabla s, \quad \mathbf{J} = \nabla s \times \nabla\zeta \qquad (2)$$

for the magnetic field **B** and the current density $\mathbf{J} = \nabla \times \mathbf{B}$, where s is the toroidal flux and θ is a multiple-valued poloidal flux. By using these variables we avoid the assumption that **B** is differentiable. However, it becomes necessary to make the hypothesis that the flux surfaces s =const. comprise a nested family of tori. That hypothesis is essential if one seeks to achieve good plasma confinement.

After renormalization, s can be employed as a radial coordinate and θ and ϕ can be introduced as invariant poloidal and toroidal angles. In this flux coordinate system we expand the magnetic field strength in a Fourier series

$$\frac{1}{B^2} = \sum B_{mn} \cos(m\theta - [n - \iota m]\phi) \qquad (3)$$

whose coefficients B_{mn} are functions of s known as the magnetic spectrum. The factor $[n - \iota m]$ in this equation indicates that the expansion is in a straight field-line coordinate system. The related formula

$$\frac{\mathbf{J}\cdot\mathbf{B}}{B^2} = p'\sum \frac{mB_{mn}}{n - \iota m} \cos(m\theta - [n - \iota m]\phi) \qquad (4)$$

for the parallel current, in which $p = p(s)$ is the pressure, can be derived from the magnetostatic equations. The small denominators $n - \iota m$ at rational values of the rotational transform ι in this representation are what shows that smooth solutions of the fully three-dimensional problem of toroidal equilibrium do not in general exist. This is why it is important to construct weak solutions of a conservation form of the partial differential equations of magnetostatics.

The NSTAB code computes fixed boundary equilibria of plasma inside a torus specified by the formula

$$r + iz = e^{iu} \sum \Delta_{mn}\, e^{-imu+inv}\,. \qquad (5)$$

In applications to stellarators it is usually sufficient to restrict the indices of the Fourier coefficients Δ_{mn} to the range $-1 \le m \le 4$, $0 \le n \le 2$. In this notation the design of optimal configurations is facilitated by a more or less direct relationship between the quantities Δ_{mn} and B_{mn} that makes it possible to achieve desirable properties of the magnetic spectrum such as quasisymmetry in an understandable way. Furthermore, the choices 1, 2 and 3 of the index m are associated respectively with helical excursion, elongation and triangularity in the shape of the plasma.

An early numerical implementation of the MHD variational principle employed the finite element method to study nonlinear stability by surveying the energy landscape. The NSTAB code is far more accurate because it describes dependence on the poloidal and toroidal angles u and v by means of a spectral representation and has a finite difference scheme in the radial coordinate s that captures discontinuities with unusually fine resolution. However, difficulties with the NSTAB expression for the potential energy at the magnetic axis make it preferable to study stability of higher order MHD modes by applying a mountain pass theorem that requires a search for multiple, or bifurcated, equilibria. A solution of the problem is found to be unstable when there is another solution with ripple in the nested flux surfaces indicative of a magnetic island or a ballooning mode. In this way one can also calculate bifurcated three-dimensional equilibria for tokamak configurations whose physics is customarily described by two-dimensional solutions that are unstable.

Predictions of linear and local stability limits on the plasma parameter $\beta = 2p/B^2$ in the Large Helical Device (LHD) stellarator at the Institute for Fusion Science (IFS) in Japan have been exceeded by experimental observations [4]. The same is true of the Wendelstein 7-AS (W7-AS) experiment in Germany [5]. These discrepancies may well be connected with issues about the convergence of numerical methods employed in the linearized theory that we attribute to the nonexistence of differentiable solutions of the equilibrium problem in three dimensions [2]. Force balance and stability may get smeared out and become lost if the partial differential equations of magnetostatics are not put in conservation form by the computer code that is applied. There is a similar problem getting the right shock jump in computational fluid dynamics.

In terms of the Maxwell stress tensor

$$\mathbf{T} = \mathbf{B}\,\mathbf{B} - (B^2/2 + p)\,\mathbf{I}\,, \qquad (6)$$

the partial differential equations describing force balance can be put in the conservation form

$$\nabla\cdot\mathbf{T} = 0, \quad \nabla\cdot\mathbf{B} = 0. \qquad (7)$$

The merit of corresponding finite difference equations is that when they are summed over a test volume they telescope into an approximate statement of force balance

$$\iint \mathbf{T}\cdot\mathbf{N}\, dS = 0 \qquad (8)$$

over the boundary. This can be understood more easily for the simplified example of the Burger's equation

$$2\,\Psi_x\,\Psi_{xx} = (\Psi_x^2)_x = \eta\,\Psi_{xxx}\,, \qquad (9)$$

subject to three boundary conditions $\Psi(-1) = \Psi(1) = 0$, $\Psi_x(-1) = 1$, which models a reversed field pinch (RFP) in slab geometry [6]. Here Ψ is the flux, Ψ_x is the principal component of the magnetic field, Ψ_{xx} is the current, and η is an artificial resistivity (cf. Fig. 1). The conservative difference scheme

$$\begin{aligned}(\Psi_{n+1} - \Psi_n)^2 - (\Psi_n - \Psi_{n-1})^2 &= \\ \eta(\Psi_{n+2} - 3\,\Psi_{n+1} + 3\,\Psi_n - \Psi_{n-1})&\end{aligned} \qquad (10)$$

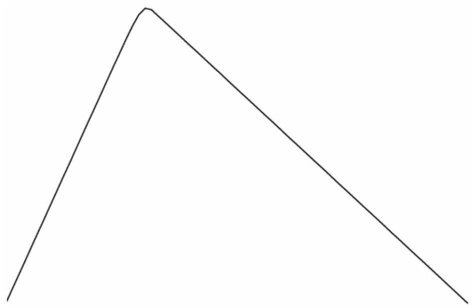

Fig. 1. Bad solution for the flux $\Psi = \Psi(x)$ in the RFP problem using a finite difference scheme that is not in conservation form. If force balance were computed accurately then the two slopes at opposite ends of the curve would have the same size and the apex would be in the middle. The slopes characterize the magnetic field on opposite sides of a current sheet, which is like a chain of magnetic islands.

computes jumps across discontinuities correctly and therefore imposes force balance across a sharp boundary that occurs at $x = 0$ in the limiting case $\eta = 0$ where the solution reduces to $\Psi = 1 - |x|$.

The nonlinear stability test implemented in the NSTAB code by calculating bifurcated equilibria gives estimates of β limits in the LHD and W7-AS experiments that are in good agreement with the measured values. Bifurcated equilibria can be obtained by triggering modes that are thought to be dangerous, or simply by iterating on the NSTAB calculations until a ballooning structure appears in the magnetic surfaces that may be wall stabilized, but otherwise appears to be unacceptable. Computations of this kind provide evidence that the NCSX and MHH2 have good physical properties, and their average β limits seem to be about 5%.

The successes of benchmarking results from NSTAB with experiments have enabled us to design QAS stellarators with two or three field periods that are attractive candidates for future experiments to validate the computational theory. These configurations, when scaled to appropriate sizes, also serve as the basis for the ARIES reactor studies. In our work, the freedom afforded by three-dimensional shaping of the plasma is used to achieve quasiaxisymmetry so that the transport of particles becomes as good as that in tokamaks. We also achieve MHD stability at high β values and modify other properties so as to make configurations more attractive as reactors. In particular, increasing depths of magnetic wells and minimizing the resonance perturbation to improve the quality of flux surfaces are important considerations in the configuration optimization. The ability to achieve good quasiaxial symmetry with only two or three field periods at low aspect ratios gives rise to the possibility of designing a compact fusion reactor. Realistic reactor designs including proper sizing of the blanket, shield and coils have been demonstrated [7]. An example showing the cross sections of the MHH2 in a bifurcated solution when the plasma pressure exceeds a critical value is given in Fig. 2.

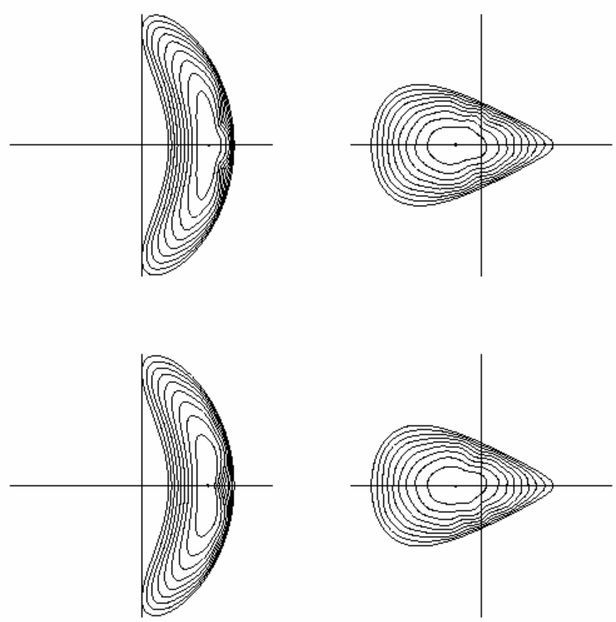

Fig. 2. Four cross sections of the flux surfaces over two field periods of a bifurcated QAS equilibrium at $\beta = 0.06$ with $p = p_0\,(1 - s^{1.1}\,)^{1.1}$ and with net current bringing the rotational transform into the interval $0.65 > \iota > 0.45$. A low order ballooning mode appears in the solution, which has an obvious asymmetry.

III. MONTE CARLO SIMULTION OF TRANSPORT

We study transport for stellarators and tokamaks by tracking particle orbits described by guiding center differential equations. A test particle model from kinetic theory is used to evaluate the confinement time, with the background fixed and collisions treated statistically. The distribution function satisfies a drift kinetic equation in which the collision operator is modeled by a second order elliptic partial differential operator acting with respect to the velocity coordinates. In the absence of the collision operator, the test particle drift kinetic equation reduces to a first order partial differential equation whose characteristics are the guiding center orbits.

The test particle model is implemented in the TRAN code by the Monte Carlo method. The test particles move on drift surfaces and jump back and forth among them following a random walk that models collisions. When a particle hits the boundary of the plasma it escapes, so the particle distribution function decays exponentially because we impose a corresponding Dirichlet boundary condition on the distribution function. In the code exponential rates of decay of expected values of carefully selected integrals of the solution of the drift kinetic equation provide estimates of the confinement time for both ions and electrons. This is one of the few methods available to study the electron case efficiently. That is important because good agreement with experimental data can be achieved by running both the ion and the electron cases, together with a mechanism to enforce quasineutrality and evaluate the electric field. Numerical simulations for tokamaks with only a radial electric field give a confinement time for the electrons significantly higher than the confinement time for the ions, so we model tokamaks by bifurcated equilibria that have three-dimensional asymmetries together with corresponding oscillations in the electrostatic potential.

The equations of magnetohydrodynamics determine the structure of the magnetic field in the plasma [2]. This enables us to express the guiding center differential equations in terms of the invariant flux coordinates s, θ and ϕ. After we renormalize θ and ϕ so that they become poloidal and toroidal angles on each flux surface, all we need to integrate the equations is a knowledge of the spectrum of Fourier coefficients B_{mn} of the magnetic field strength. The mathematical analysis suffices both to study thermal confinement and to track the orbits of α particles.

When the displacement current is retained in Maxwell's equations, quasineutrality introduces fluctuations of the electric potential Φ along the magnetic lines in three dimensions. The fluctuations are associated with turbulence that leads to significant transport. This complicated time-dependent behavior must be taken into account in our model, which is based on static fields. We also need a new condition to determine the electric field when it is left out of Ampere's law. The right thing to do is to consider quasineutrality as an equation for the electric potential Φ because that follows from dropping the highest derivatives in Poisson's equation

$$\lambda^2 \Delta\Phi = n_e - n_i \quad , \tag{11}$$

where λ is the Debye length and n_e and n_i are the number densities of the electrons and the ions. The theory of singular perturbations suggests that when the left side is neglected, the resulting requirement

$$n_e(\Phi) - n_i(\Phi) = 0 \tag{12}$$

must determine the electric field, even though Φ no longer appears explicitly.

We bring the confinement time τ_e of the electrons down to the confinement time τ_i of the ions by solving the quasineutrality and ambipolarity equations $n_e = n_i$, $\tau_e = \tau_i$. The algorithm implemented in the TRAN code requires that the Fourier coefficients of the densities n_i and n_e of the ions and electrons coincide. The electric potential Φ is expanded as a Fourier series

$$\Phi = \sum P_{mn} \cos(m\theta - n\phi) \tag{13}$$

in the invariant poloidal and toroidal angles θ and ϕ with coefficients P_{mn}, measured in units of the temperature T, that depend on the flux s. The charge separation is expanded in a similar Fourier series

$$n_e - n_i = \sum C_{mn} \cos(m\theta - n\phi) \tag{14}$$

after dividing out the total number of electrons or ions. Calculation of the Fourier coefficients C_{mn} is accomplished by using the Monte Carlo method to estimate the expected values of appropriate trigonometric functions.

We have to determine the coefficients P_{mn} so that $C_{mn} = 0$. Charge separation gives rise to electrostatic restoring forces, and therefore it is reasonable to expect wells and hills in the electric potential to correct for similar oscillations in n_e - n_i. There is a more or less direct relationship between the two arrays P_{mn} and C_{mn}, so we define an iteration driving the coefficients C_{mn} towards zero by choosing a relaxation factor ε and putting

$$P_{mn}^{l+1} = P_{mn}^{l} + \varepsilon C_{mn}^{l} \quad . \tag{15}$$

Numerical calculations with this algorithm show that coefficients P_{mn} of relatively small terms in Φ that depend on the poloidal and toroidal angles θ and ϕ simulate anomalous transport of electrons by reducing τ_e more than τ_i.

The algorithm we have presented does not determine P_{00} because C_{00} is normalized to be zero. But because the TRAN code computes both the ion and the electron confinement times, we can solve the ambipolarity equation $\tau_e = \tau_i$ to obtain a physically relevant value of P_{00}. For stellarators that is easy because an increase of P_{00} increases the ion confinement time while leaving the electron confinement time largely unchanged (cf. Fig. 3). For tokamaks there is an analogous theory of transport that involves bifurcated equilibria in three dimensions and agrees well with observations, but it is too complicated to be discussed in detail here.

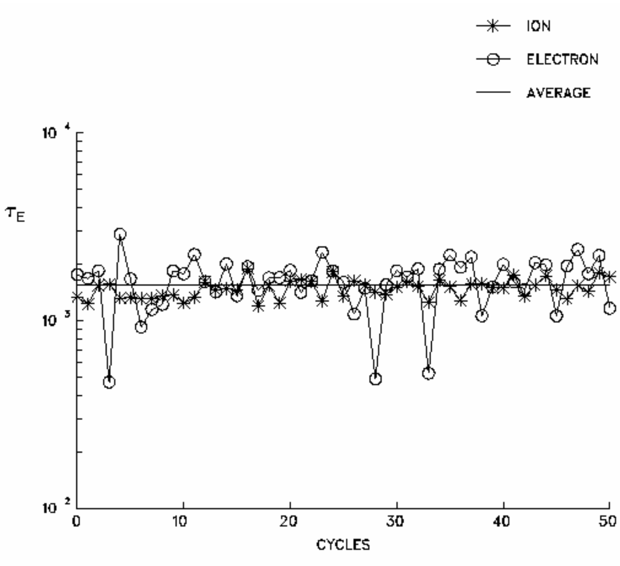

Fig. 3. Iterations to quasineutrality in a Monte Carlo computation of the energy confinement time τ_E, measured in milliseconds, for an NCSX configuration with major radius 7.2 m and plasma radius 1.6 m at average T = 15 keV, $n = 2 \cdot 10^{14}$ cm^{-3}, and B = 6.5 T. The magnetic spectrum has good quasiaxial symmetry, and the radial electric field rises to a potential level twice as big as the temperature.

IV. COILS AND REACTORS

Stellarators with quasihelical symmetry (QHS) that have four or five field periods have been used to design reactors, but the aspect ratio is big, which makes maintenance hard. The QAS concept leads us to study the MHH2 with two field periods and the NCSX with three, for which experimental data may become available soon. Our model of an NCSX reactor has aspect ratio 4.5, which leads to a relatively low loss rate of α particles, and reversed shear helps with the suppression of islands at dangerous resonances. The NSTAB and TRAN computer codes have been applied to the analysis of this configuration.

In the design of a stellarator experiment or reactor, the geometry of the coils becomes an issue as important as the physics of equilibrium, stability and transport. If the coils are smooth enough, erroneous high order harmonics of the magnetic field may be suppressed so that the flux surfaces become robust and there is structural stability in the system. When the coils are found by means of the Biot-Savart law from a fixed boundary equilibrium optimized for confinement, the choice of a control surface on which corresponding filaments are drawn becomes a problem, as does also the representation there of a distribution of surface current. Adequate space is needed between the control surface and the separatrix of the plasma, but the coils cannot be located so far away that they become excessively twisted.

The NSTAB and TRAN computer codes, combined with numerical implementation of the Biot-Savart law, have enabled us to design an MHH2 fusion reactor with just 8 modular coils not so unlike those in a tokamak (cf. Table 1 for the geometric coefficients defining the plasma boundary and Fig. 4 for a perspective view of the plasma and coil shapes). The major radius is 6.5 m and the average plasma radius is 2.6 m. Filaments specifying the coils have a distance from the separatrix exceeding 145 cm. The coil structures have a radial depth of 50 cm and a poloidal breadth of 75 cm. The magnetic field strength inside the plasma is 5 T and the maximum field on the coils is 14 T. Excellent quasiaxial symmetry produces tokamak-like confinement, and efficient conditions with average ion temperature T = 16 keV and electron density $n = 1.25 \cdot 10^{14}$ cm^{-3} are achieved. There are minimal forces between the coils because of the way they are defined by a distribution of current on a perfectly conducting control surface. But the prompt loss of α particles exceeds 10%, which is high, so that calls for more research.

The QAS stellarator with eight coils and small size is attractive for a proof of principle plasma physics

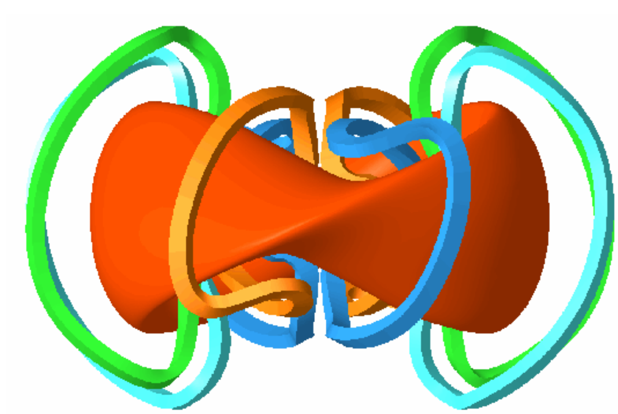

Fig. 4. Diagram of a two field period reactor in a magnetic field given by the Biot-Savart law. Eight only moderately twisted modular coils produce robust flux surfaces that do not deteriorate when changes are made in the vertical and toroidal fields. There is good access between the coils for maintenance. (Courtesy of Tak-Kuen Mau and Tsueren Wang of UCSD.)

experiment, but for a reactor one must also look at configurations with more coils, more field periods, larger sizes, and better quasisymmetry to minimize the losses of α particles. Reactors having NCSX-like plasma can be designed with eighteen modular coils, and excellent quasiaxial symmetry can be attained, although the coils are visibly twisted [8]. Also QHS reactors with 4 periods and 32 modular coils have been found [2]. These solutions of the problem have a prompt loss of α particles under 10%.

Table 1. Fourier coefficients Δ_{mn} of the surface of a plasma defined by Eq. 5. The prompt loss of α particles is below 10% when this configuration is run as a tokamak-stellarator hybrid reactor with major radius 9 m.

m	n	Δ_{mn}
-1	-1	0.200
-1	0	0.140
0	0	1.000
0	1	0.026
0	2	0.006
0	4	-0.001
1	-2	0.005
1	-1	0.070
1	0	2.500
1	1	0.094
1	2	-0.006
1	3	-0.010
1	4	-0.001
2	-1	0.006
2	0	-0.100
2	1	-0.390
2	2	-0.076
2	3	-0.005
2	4	0.002
3	1	0.010
3	2	0.077
3	3	0.028
3	4	0.006
4	-1	-0.002
4	0	0.015
4	1	0.030
4	2	-0.020
4	3	-0.010
4	4	-0.003

Observations of unexpectedly high β limits in the LHD and W7-AS experiments have raised hopes of finding a configuration with stellarator stability and tokamak transport. The three-dimensional computer codes we have described that were run for the ARIES project in this search could also be applied to the question of designing an advanced tokamak reactor.

ACKNOWLEDGMENTS

This work has been supported by the United States Department of Energy under Grants DE-FG02-86ER53223 and DE-AC02-76-CHO-3073.

REFERENCES

[1] G.H. NEILSON, M. ZARNSTORFF, L. P. KU, et al., “Physics Considerations in the Design of NCSX,” *Proc. of 19th International Atomic Energy Agency Fusion Energy Conference*, IAEA-CN-94/IC-1, Lyon, France (2002).

[2] P. GARABEDIAN, “Three-dimensional Tokamak Equilibria and Stellarators with Two-dimensional Magnetic Symmetry,” *Plasma Phys. Control. Fusion,* **39B**, 129 (1997).

[3] P. GARABEDIAN and L. P. KU, “Quasiaxially Symmetric Stellarators with Three Field Periods,” *Phys. of Plasmas*, **6**, 645 (1999).

[4] K. WATANABE, H. YAMADA and A. KOMORI, “MHD Properties of High Beta Plasma and Recent Results in LHD Experiments,” *Proc. of the U.S.-Japan Workshop and 21st COE Symposium on “New Approach in Plasma Confinement Experiment in Helical Systems,”* Kyoto University, Kyoto, Japan (2004).

[5] A. WELLER, J. GEIGER, A. WERNER, et al., “Experiments Close to the Beta-Limit in W7-AS,” *Plasma Phys. Control. Fusion* **45**, A285 (2003).

[6] F. BAUER, P. GARABEDIAN, D. KORN and A. JAMESON, *Supercritical Wing Sections II*, Springer-Verlag, New York (1975).

[7] L. EL-GUEBALY, R. RAFFRAY, S. MALANG, et al., “Benefits of Radial Build Minimization and Requirements Imposed on ARIES Compact Stellarator Design,” *Proc. of ANS 16th Topical Meeting on the Technology of Fusion Energy*, Madison, Wisconsin (2004).

[8] L. P. KU et al., “A Compact Quasi-axisymmetric Stellarator Reactor,” *Proc. of 20th IEEE/NPSS Symposium on Fusion Engineering*, San Diego, California (2003).