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EXAMINATION OF STABILITY CALCULATIONS FOR LIQUID METAL FILM FLOWS IN A COPLANAR MAGNETIC FIELD

Использование течений тонких пленок жидкого металла в копланарном магнитном поле для защиты поверхностей элементов термоядерного реактора, контактирующих с плазмой, представляет определенный интерес, в связи с этим были проведены различные исследования, предметом которых являлись развитие профиля (высоты) пленки и ее устойчивость. Первые работы в этом направлении с использованием одномерных моделей были выполнены Т. Н. Аитовым с сотрудниками в конце 80-х годов. Очевидные недостатки этих работ побудили К. Ляо и его коллег заняться уточнением этих моделей и провести более глубокий их анализ. Хотя и в работах Т. Н. Аитова были обнаружены ошибки, анализ, проведенный нами, позволяет сделать вывод, что его метод, по существу, корректен. Уточнение, осуществленное Ляо, раскрыло упомянутые ошибки, но критика была направлена по не совсем правильному адресу. В нашей работе ошибки Аитова устранены без уклонения от его оригинального подхода и предложены соображения относительно пределов применимости известных расчетов устойчивости исходя из более совершенных развитых двухмерных профилей скорости.

1. REVIEW OF AITOV'S METHOD

Aitov's equations predict the film height ($\hat{y}$) [1, eqn. 2.11] and stability properties [2, eqn. 2.14] of a thin film of liquid metal flowing down ($\hat{x}$) an inclined, electrically non-conducting channel in a coplanar magnetic field ($B_{\hat{y}}$). They are derived from the basic set of MHD equations; including the Navier—Stokes equation, continuity equation, current conservation equation and Ohm's law; by averaging over the channel width ($\hat{z}$) assuming that the velocity conforms to a strong Hartmann profile $V(x,y,z)=V(x,y)\cdot\left(1-\exp\left[\mathrm{Ha}\left(\left|\frac{z}{b}\right|-\frac{1}{2}\right)\right]\right)$. Here Ha is the Hartmann number and b is defined as the channel width where the sidewalls are located at $z=\pm b/2$. Figure 1 shows the channel geometry.

The $\hat{y}$ component of the Navier—Stokes equation is simplified by the neglect of the inertial and time dependent terms due to the «shallow water approximation». Viscous terms are also neglected in the $\hat{y}$ equation. The simplified equations, given in [2, eqns. 1.18 ... 1.23], are then combined by solving the $\hat{y}$ component of the Navier—Stokes equation for the pressure and substituting this into the $\hat{x}$ equation [2, eqn. 1.26]:

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{\cos\theta}{\mathrm{Fr}^2}\frac{\partial h}{\partial x}+\frac{\mathrm{We}}{\mathrm{Fr}^2}\frac{\partial^3 h}{\partial x^3}+\frac{1}{\mathrm{Re}}\frac{\partial^2 u}{\partial y^2}-$$

$$-\frac{2\mathrm{Ha}\,\beta^2}{\mathrm{Re}}u+\frac{\sin\theta}{\mathrm{Fr}^2}. \qquad (1)$$

Here u and v are the normalized components of the velocity in the $\hat{x}$ and $\hat{y}$ directions; h is the normalized film height; and Re, Fr, and We are respectively the Reynolds, Froude, and Weber numbers. The fully developed film height h_* is used as the characteristic length (except for

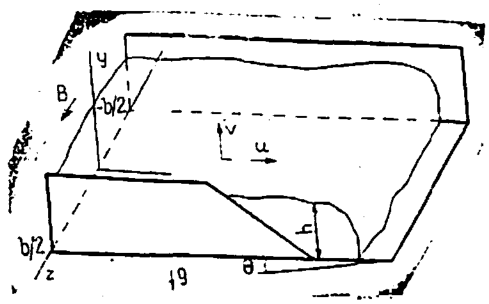

Fig. 1. Channel geometry.

Рис. 1. К постановке задачи.

the definition of the Hartmann number where b is used), and the fully developed average velocity u_* is taken as the characteristic velocity. The time variable is normalized with h_*/u_* and β is the length ratio h_*/b.

The 1-D film height evolution (or developing film) equation is obtained by omitting the time dependent term of eqn. 1 and integrating over the film height, assuming a specified velocity distribution in y. The result is a 1st order differential equation for the film height [1, eqn. 2.11] as a function of the downstream x-coordinate. Note that the We term is neglected since it contains a higher order derivative.

The stability analysis [2] is performed by perturbing the equilibrium solution of the fully developed version of eqn. 1 by a small amount. This perturbation is given in terms of a stream function ψ and all terms quadratic in the small perturbations are discarded. The stream function perturbation is then assumed to be of the form $\psi(x, y, t) = \Phi(y)\exp[ik(x-ct)]$ where k stands for the wave number and c is the phase velocity of the disturbance.* The equations decribing the perturbation [2, eqn. 2.3...2.6]

$$\Phi' ik(U-c) - ik\Phi\frac{\partial U}{\partial y} = \frac{1}{\mathrm{Re}}\Phi''' - \frac{2\,\mathrm{Ha}\,\beta^2}{\mathrm{Re}}\Phi' - \frac{ik}{\mathrm{Fr}^2}\left(\frac{\Phi(1)}{c-U(1)}\right)(\cos\theta + k^2\,\mathrm{We}), \tag{2}$$

$$\Phi(0) = 0, \tag{3}$$

$$\Phi'(0) = 0, \tag{4}$$

$$\Phi''(1) = \frac{\Phi(1)}{U(1)-c}\,\frac{\partial^2 U(1)}{\partial y^2} \tag{5}$$

where the primes denote differentiation with respect to y. Boundary conditions (3) and (4) follow from the no-slip and impenetrable wall conditions for u and v on the channel substrate, and condition (5) results from the no shear stress condition on the free surface. Here $U(y)$ denotes the fully developed equilibrium velocity profile which is being perturbed.

For long waves ($k \ll 1$), the solution to eqns. (2...5) can be obtained by solving successive approximations, where the wave function and phase velocity are expanded:

$$\Phi = \Phi_0 + k\Phi_1 + k^2\Phi_2 + \ldots, \tag{6}$$

$$c = c_0 + kc_1 + k^2 c_2 + \ldots \tag{7}$$

and each component of Φ and c is solved for keeping only terms that order in k. Aitov solved for the zeroth and first order terms and used

* This equation is changed from [2, eqn. 2.2] by a minus sign; this is most likely a typographical error.

the condition that any imaginary component of c must be negative or the perturbation grows in time and the equilibrium is unstable. It is worth noting that even if the «shallow water approximation» was not used, the same equations for the zeroth and first order cases would result.

2. LIAO'S CONTRIBUTION

In [3], Liao examined Aitov's work and found several errors in it; in particular a discrepancy between the velocity profile used in the developing film analysis and that perturbed in the stability analysis, and the failure of the solutions for Φ and c to reduce to the classical hydrodynamic result in the limit $B \to 0$. In order to eliminate these inconsistencies, Liao repeated both the developing film and stability calculations using Aitov's methodology, but employing a parabolic velocity profile like [1, eqn. 2.9] and a normalization based on the channel inlet conditions. His developing film model is the same as Aitov's but the stability model differs, and does reduce to the classical solution. This is not wholly unexpected since the parabolic velocity profile *is* the classical hydrodynamic solution to begin with.

Liao also investigated the effect of the film contact with an energetic plasma. To this end, he performed a calculation of the force exerted by normal incident plasma ions and the pulling action of the electrostatic plasma sheath. Characterizing the sum of these opposing forces as a pressure (P_a) acting normal to the surface. P_a is added to the normal stress condition which contributes the term $\partial P_a/\partial x$ to the right-hand side of eqn. 1, where $P_a = P_a/\rho u_*^2$ and u_* is the characteristic velocity. This pressure affects the film only in areas of rapid spatial change of the plasma parameters, like the flowing film's entrance to or exit from the plasma beam. Liao concluded that the plasma has little effect on either the film development or stability, and he discarded it in subsequent calculations. Investigation of this effect was given also in [5] where, for high density liquid metals like gallium, results similar to Liao's were presented.

Another point evident from Liao's calculations, although not discussed in [3], is that the increase in momentum of the ions as they accelerate through the plasma sheath is *equal* to the pulling force of the plasma sheath. Using Liao's results we find

$$|P_{n_w} - P_{n_s}| = |P_{E_w}| = 2n_s k_B T \qquad (8)$$

where P_{n_w} and P_{n_s} are the effective pressures of the ions at the film surface and the sheath edge; P_{E_w} is the «negative pressure» due to the sheath pulling action; and n_s, k_B, and T are the ion density in the sheath, the Boltzmann constant, and the electron/ion temperature respectively. Since the increased momentum of the ions in the sheath is exactly balanced by the electrostatic pulling action, the use of the presheath values of the ion momentum in calculating the plasma's force on the surface is in order. This conclusion was intuited by Murav'ev based on momentum conservation [4].

Due to the grazing angle of incidence of the magnetic field in the pre-sheath, and thus the ion guiding centers as well, a large momentum flux in the $\hat{z}$ direction is to be expected. This phenomenon may considerably alter the surface shape profile in $\hat{z}$ making the assumption of a flat surface over the film width, implicit in all these models, not applicable. It is clear that the treatment of the plasma as a simple pressure does not address this issue

3. RECALCULATION OF AITOV'S EQUATIONS

Although there are indeed inconsistencies in Aitov's work, it is the conclusion of the authors that his methodology is essentially correct. Based on this conclusion, a rederivation of his results is performed using the self-consistent velocity profile for both the developing film and stability models. Liao's concern that the flowrate of this profile is determined by internal parameters (Ha, β, and θ) instead of the external supply is unfounded. Aitov's choices (in [2]) of characteristic length and velocity are the fully developed values of these variables, both of which are problem-dependent. These choices make the subsequent stability equations simpler to derive. But the dependence of U on q (the externally controlled flowrate per unit width) is still present, even though it is not explicitly stated in [2]. The equilibrium velocity, $\bar{u}$, is found by solving the dimensional, fully-developed version of eqn. 1.* Averaging $\bar{u}$ over the film height gives [2, eqn. 1.27]

$$u_* = \frac{g h_*^2}{2\,\mathrm{Ha}\,\beta^2 \nu} \sin\theta \left(1 - \frac{\mathrm{th}\left[\sqrt{2\,\mathrm{Ha}\,\beta^2}\right]}{\sqrt{2\,\mathrm{Ha}\,\beta^2}} \right) \tag{9}$$

where g and ν are the acceleration of gravity and the kinematic viscosity. This, however, is only one equation for two unknowns. The second necessary equation is simply

$$u_* h_* = q. \tag{10}$$

Thus the fact that the flowrate is constant is implicitly included in the selection of the normalization. Unfortunately, since the length h_* is included in the definition of β, the above equations are transcendental. This makes the analysis of specific dimensional problems more difficult than if constant inlet values of h and u were used.

In order to reconcile the fact that Aitov's developing flow calculations employed a different velocity profile than that discussed above, recalculation of Aitov's model using the self-consistent velocity is in order. This velocity profile, as mentioned above, is determined from the dimensional version of eqn. 1 and is renormalized with the constant flowrate condition:

$$u(x, y) = \frac{q}{h(x)\left(1 - \dfrac{\mathrm{th}[mh(x)]}{mh(x)}\right)} \left(\mathrm{th}[mh(x)]\,\mathrm{sh}[my] - \mathrm{ch}[my] + 1\right). \tag{11}$$

Here m is defined to be $\dfrac{\sqrt{2\,\mathrm{Ha}}}{b}$ for convenience. This equation is presented in dimensional form in order to match the development in [1]. Following the procedure outlined in section 1, the developing film height model is rederived and non-dimensionalized with the characteristic quantities u_{in} and h_{in} in order to facilitate calculation. This new model is

$$\mathrm{d}h/\mathrm{d}x = \frac{\dfrac{h \sin\theta}{\mathrm{Fr}^2} - \dfrac{m_*^2}{\mathrm{Re}}\left(1 + \dfrac{\mathrm{th}[m_* h]}{m_* h - \mathrm{th}[m_* h]}\right)}{\dfrac{h\cos\theta}{\mathrm{Fr}^2} - \dfrac{m_*^2\,\mathrm{sh}[m_* h]\left(2 - 2\,\mathrm{ch}[2m_* h] + 2m_*^2 h^2 + m_* h\,\mathrm{sh}[2m_* h]\right)}{2\left(m_* h\,\mathrm{ch}[m_* h] - \mathrm{sh}[m_* h]\right)^3}} \tag{12}$$

* For clarity, the dimensionless equilibrium velocity used in eqns. 2 ... 5 is $U = \bar{u}/u_*$.

where $m^2 = 2\,\mathrm{Ha}\,\beta^2$, and the definitions of the dimensionless quantities Fr, Re, Ha, and β are taken from [2] instead of [1] as a more appropriate choice. It is noted that eqn. 12 reduces to the form given in [1, eqn. 2.11] in the limit $m_* \to 0$, which is the classical hydrodynamic solution.

Liao's claim that the stability solutions of Aitov do not reduce to the classical hydrodynamic solutions given by Yih [6] is true. Even though Aitov's use of the self-consistent velocity profile is the correct procedure, his solutions for Φ and c are incorrect. However, this is only a mistake in solving the equations and can be carrected to give

$$\Phi_0 = A_0 \left(\frac{\mathrm{ch}[m_* y] - 1}{m^2} \right); \tag{13}$$

$$c_0 = \frac{\mathrm{th}^2[m_*]}{1 - (\mathrm{th}[m_*])/m_*}; \tag{14}$$

$$c_1 = -i\left(\mathrm{ctg}\,\theta + \frac{k^2\,\mathrm{We}}{\sin\theta} \right) +$$

$$+ i\,\mathrm{Re}\left(\frac{\mathrm{sh}[m_*](4m_* + 2m_*\,\mathrm{ch}[2m_*] - 3\,\mathrm{sh}[2m_*])}{4\,\mathrm{ch}^3[m_*](m_*\,\mathrm{ch}[m_*] - \mathrm{sh}[m_*])^2} \right). \tag{15}$$

All of these equations reduce to the classical result given in Yih for the limit $m_* \to 0$.

Another work by Kirillina [7] expanded the Aitov method to consider a small component of the magnetic field lying with arbitrary orientation in the xy-plane. However, simplifying her equations to the case of a purely coplanar field critical Reynolds number for negative $Im(c_1)$, and hence for linear stability, reduces to $\mathrm{Re}_{cr} = \frac{80}{101}\left(\mathrm{ctg}\,\theta + \frac{k^2\,\mathrm{We}}{\sin\theta} \right)$ in the non-magnetic case instead of the $\frac{5}{6}(\ldots)$ predicted by Yih. Although it does not appear that Kirillina's solution is completely correct, it is very close and her conclusions about the stabilizing effect of the small component of the magnetic field in the xy-plane should not be discounted.

Fig. 2 compares the Aitov/Morley and Liao solutions for a typical chute inclination angle. The figure shows curves of neutral stability as a function of the modified Hartmann number, m_*. Our results predict a more stable film than the Liao formulation This may be due to the increased flatness of the Aitov velocity profile at the surface of the film. There is less energy available in the flat profile to fuel the instability.

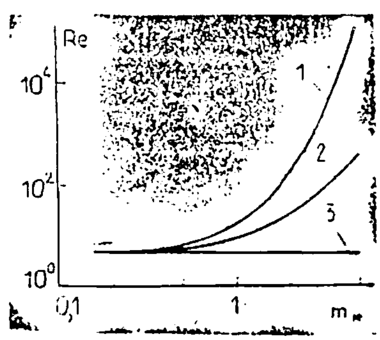

Fig. 2. Critical Reynolds numbers for instability at $\theta = 10°$ vs $m_* = (2\mathrm{Ha})^{1/2} h_*/b$, h_* being the fully developed film height, b the channel width; *1* — Aitov/Morley, *2* — Liao, and *3* — Yih ($B = 0$). Unstable region — above the curves.

Рис. 2. Зависимости критического числа Рейнольдса для неустойчивости при $\theta = 10°$ от $m_* = (2\mathrm{Ha})^{1/2} h_*/b$ (h_* — высота полностью развитой пленки, b — ширина канала): *1* — Аитов/Морлей; *2* — Ляо; *3* — Йих ($B = 0$). Область неустойчивости — над кривыми.

Liao suggests it is inappropriate to retain $\dfrac{k^2\,\mathrm{We}}{\sin\theta}$ due to the presence of k^2 in the first order approximation. Retention of this term however, is justified by the fact that We appear nowhere else in the stream function equations and thus its inclusion can only improve the accuracy of the solution [6, p. 325]. For long waves and fusion conditions, the term contributes very little and it is the presence of the Hartmann layer, as Aitov correctly points out, that is the major stabilizing force.

4. ACCURACY OF 1-D METHODS

The averaging process by which the original equations are simplified relies on the assumption that the velocity conforms to a Hartmann type profile across the width of the channel. This assumption breaks down near the channel substrate and near the surface due to the need for the induced electrical currents to close through the Hartmann layer. This phenomenon causes the formation of Shercliff side-layers in MHD duct flow. A similar layer must form on the surface of a film if the currents are to turn, perserving $J_n=0$ (surface normal current density). Thus, close the surface there is little current perpendicular to the z-directed magnetic field, and therefore little magnetic force. The result is the formation of a quasi-parabolic side-layer near the film surface. An example shown in Fig. 3, computed from a 2-D fully developed MHD film flow program, shows a very peaked surface side-layer. Even though the Aitov velocity profile matches very closely the film behaviour in the majority of the channel, at the surface the 1-D model fails (Fig. 4). This peaked profile, owing to the logic at the end of the previous section, will be much more unstable than the flat Aitov profile. It is possible that Liao's solution will be closer to predicting the actual stability due to the parabolic profile assumed in its development. However, for magnetic fields not exactly coplanar, the surface sidelayer is quickly decimated as the angle between B and z increases in magnitude and a flat profile is restored in some cases.

The accuracy of either result is suspect due to the deviation of the velocity profile models from the actual profile. Experimental corroboration

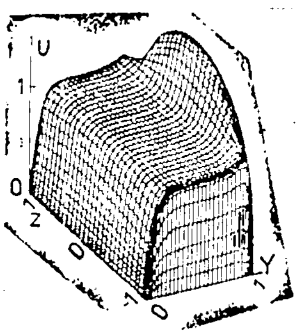

Fig. 3. 2D velocity profile at $\mathrm{Ha}=5\cdot10^4$ and $\beta=0.05$.

Рис. 3. Двухмерный профиль скорости при $\mathrm{Ha}=5\cdot10^4$ и $\beta=h/b=0.05$.

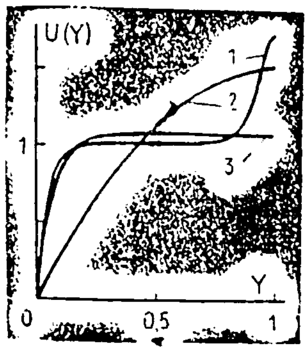

Fig. 4. Velocity profiles at channel centerline: *1* — 2D fully developed flow, *2* — 1D parabolic, and *3* — 1D Aitov.

Рис. 4. Профили скорости вдоль оси канала: *1* — двухмерное полностью развитое течение; *2* — одномерный параболический профиль; *3* — одномерный профиль по Айтову.

exists for the developing flow models, but little data is available to verify the stability analysis results. An analysis attempting to address this deviation is currently under development by the authors.

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