I. INTRODUCTION

1.1 Frequency Domain

In principle, the voltages and currents in analog circuits are arbitrary functions of time (we call them signals or waveforms). Analytical analysis of the circuit response to an arbitrary input waveform is difficult and requires solution to a set of differential equations. Even numerical analysis becomes difficult when there are a lot of circuit elements. Fortunately, there are ways to find the response of a linear circuit to time-dependent signal. These approaches are based on the following observations:

1. For circuits driven by sinusoidal sources, the forced response of the state variables (currents and voltages) are all sinusoidal functions with the same frequency as the source.

   This is derived from the mathematical properties of sinusoidal functions. Forced response of a set of linear differential equations (circuit equations) to a sinusoidal function is a sinusoidal function. This property leads to special analysis tools for AC circuits using “phasors,” or using Fourier transform. AC steady-state analysis of linear circuits are covered in ECE35/45. When we use phasors, the circuit equation do not contain time anymore, but they include frequency $\omega$. As such, this is usually called analysis in “frequency-domain” to differentiate that from “time-domain” analysis where we solve the differential equation to find the circuit response.

2. Any arbitrary but periodic signal can be written as a sum of sinusoidal functions using Fourier series expansion.

   For example, a square wave with period $T$ or frequency $\omega_0 = 2\pi f = (2\pi)/T$ and amplitude $V_m$ can be written as:

   \[ v(t) = \frac{4V_m}{\pi} \left[ \sin(\omega_0 t) + \frac{1}{3} \sin(3\omega_0 t) + \frac{1}{5} \sin(5\omega_0 t) + \ldots \right] \]

   Signals with frequencies $n\omega_0$ ($n$ integer) are called harmonics of the fundamental frequency, $\omega_0$. In general the amplitude of higher harmonics become smaller as $n$ become larger. The idea of decomposition of a periodic function to a sum of sinusoidal functions can be extended to an arbitrary temporal function by using Fourier integrals. As such, in principle, any function of time can be written as a sum of (or an integral of) sinusoidal functions.

3. Proportionality and superposition principles state that the response of a linear circuit to a linear combination of sources is equivalent to the linear combination of the circuit response to each individual source.
Basically, in a circuit with several independent sources, the value of any state variable equals to the algebraic sum of the individual contributions from each independent source. So, in a circuit with a time-dependent source, we can use Fourier series decomposition and replace the source with a linear combination of several sinusoidal sources. We can then find the response of the circuit to each sinusoidal source and then use proportionality and superposition to find the response to the time-dependent source.

For example, suppose we want to have a circuit driven by a source that can be decomposed into \( v_i(t) = A \cos(100t) + B \cos(300t) \). We want to know the voltage across an element, \( v_o(t) \). We solve the circuit with the source \( \cos(100t) \) and find the voltage across the element interest, suppose \( \alpha \cos(100t + \phi_\alpha) \). We then repeat the analysis with a source \( \cos(300t) \) and find the voltage across the element interest, suppose \( \beta \cos(300t + \phi_\beta) \). The response of the circuit to \( v_i(t) = A \cos(100t) + B \cos(300t) \), then is \( v_o(t) = A\alpha \cos(100t + \phi_\alpha) + B\beta \cos(300t + \phi_\beta) \).

The problem is actually much simpler than the example above. In principle, solution of AC steady-state circuit is simple and we typically find the response the circuit with frequency, \( \omega \), as a parameter. We can then construct the response by replacing \( \omega \) with frequencies of interest in the response equation (e.g., set \( \omega = 100 \) and \( 300 \) in the above example). Another major simplification arises when the circuit response is frequency independent. In that case, the circuit response can be directly applied to any time-dependent function. For example, in the above example, if the circuit response to \( \cos(100t) \) and \( \cos(300t) \) sources were, respectively, \( \alpha \cos(100t) \) and \( \alpha \cos(300t) \) (frequency independent), then the circuit response is simply: \( v_o(t) = \alpha v_i(t) \).

Therefore, we focus on circuits driven by sinusoidal sources. We solve these circuits in frequency domain. We try to find circuit parameters with frequency \( \omega \) as a parameter to facilitate construction of response to an arbitrary function of time.

There are several ways to solve the circuit in frequency domain, all having the same mathematical foundation. We can use phasors (which are really Fourier Transforms). Or, we can use complex frequency domain which is sometimes called “s-domain” \( s = \sigma + j\omega \). In junior level courses and beyond, you will probably use complex frequency domain mainly. Circuit analysis with phasors is sufficient for the work we do in this class (to convert from phasors to s-domain), simply replace \( j\omega \) with \( s \) and \( -\omega^2 \) with \( s^2 \).

Analysis in frequency domain is straight-forward. Resistors, capacitors, and inductors are replaced by impedances, \( Z \): \( Z = R \) for a resistor, \( Z = 1/(j\omega C) \) for a capacitor and \( Z = j\omega L \) for an inductor. Impedances obey Ohm’s Law: \( V = ZI \). Thus, with impedances the circuit reduces to a “resistive” circuit and all analysis techniques of resistive circuits (node-voltage method, mesh-current method, Thevenin Theorem, etc.) apply. The only difference is that analysis is performed using complex variables.
1.2 Circuit Components

It is not practical to design a complete circuit as a whole from scratch. It is usually much easier to break the circuit into components and design and analyze each component separately. In this manner we can design “building blocks” (such as amplifiers, filters, etc.) that can be used in a variety of devices. A typical analog circuit is composed of a “source,” a “load,” (both are “two-terminal networks”) and one or several “two-port networks.” Note that these components “communicate” with each other only through the attaching wires, *i.e.*, through currents and voltages.

1.3 Two-Terminal Networks

Let us first consider two-terminal networks, boxes with only two wires. In linear circuit theory, we have nine ideal circuit elements, denoted by their $i - v$ characteristics:

- **Resistor:** $v = Ri$
- **Capacitor:** $i = c \frac{dv}{dt}$ or $V = \frac{1}{j\omega C} I$
- **Inductor:** $v = L \frac{di}{dt}$ or $V = j\omega LI$

**Independent voltage source:** $v = v_s = \text{const.}$ for any current

**Independent current source:** $i = i_s = \text{const.}$ for any voltage

and four controlled sources: voltage-controlled voltage source, current-controlled voltage source (similar to independent voltage source but with source strength depending on voltage or current on another element in the circuit) and voltage-controlled current source, current-controlled current source.

It is essential to remember that these “ideal” circuit theory elements do NOT represent physical devices, rather they are idealized elements, “cooked” up to simplify the analysis (because their voltage-current relationship is linear). Physical elements that we encountered in the real world can only be modeled with one of these ideal elements within a certain range of parameters and within a certain accuracy. For example take a resistor in the lab. At
high enough frequencies, it will exhibit capacitance (*i.e.* its “resistance” drops as frequency increases). At high enough current, when the resistor is hot enough, the ratio of \( v/i \) is not linear anymore. So, an “ideal” resistor used in circuit theory is NOT a physical device. Rather, a “real” resistor in the lab can be approximated by an “ideal” resistor only for a range of current or voltage (typically rated by its maximum power), a range of frequencies, and even a range of environmental conditions (temperature, humidity, *etc.*) only when the voltage across this “real” resistor is directly proportional to the current flowing through, *i.e.*, the element \( i - v \) characteristics equation is \( v = Ri \).

We can take this observation one step further: Any two-terminal network (a box with two wires) whose voltage is directly proportional to the current flowing through, *i.e.*, the element \( i - v \) characteristics equation is \( v = Ri \), can be modeled as an “ideal” resistor. This means, that we can take this box out of the circuit and a replace it with a resistor and the response of the rest circuit does not change. Most importantly, we do NOT need to know what is inside the box, the only parameters we need to know is its resistance value.

Similarly, if we have a black box whose voltage is a constant for all currents, it can be modeled as an independent voltage source (without any knowledge of what is inside the box). You actually have been doing this in the lab, modeling the power supply (which includes many transistors, diodes, resistors, capacitors, ICs) with an independent voltage source.

Another example are inductors we use in the lab. These “real” inductors are typically manufactured by tightly wrapping a wire around a ferrite core (to boost its inductance). Because the wire itself has a resistance, at low frequencies a good model for this “real” inductor is an “ideal” resistor (resistance of the wire) in series with an “ideal” inductor. Only if the resistance of the wire is small compared to other resistances in the rest of the circuit, our “real” inductor can be modeled as an “ideal” inductor. Otherwise, we need to account for the resistance of the “real” inductor (we will explore this in Lab 2).

The bottom line is that we need to consider the \( i - v \) characteristics of a two-terminal network and if they follow the \( i - v \) characteristics of one of the five elements above, we can model the two-terminal network with the corresponding element.

With these observations in hand, we can now proceed to the general case of any two-terminal network (a box with two wires) by utilizing the Thevenin Theorem.
1.4 Thevenin Theorem and Thevenin or Norton Equivalents

We know from linear circuit theory that the $IV$ characteristics of a two-terminal network is in the form of (using active sign convention):

$$V = V_T - Z_T I; \quad Z_T = Z_n; \quad I_n Z_n = V_T$$

which is similar to the $IV$ characteristics of the Thevenin or Norton forms shown above. Therefore, any two-terminal network can be modeled by two parameters only, $Z_T$ and $V_T$ (or $I_n$ and $Z_n$). We can replace our black box with either of the Thevenin or Norton equivalent circuits above and the response of the rest of the circuit does not change.

An important corollary to the Thevenin Theorem is that if a two-terminal network does not include an “independent source” it can be reduced to a single “impedance” (even if it includes dependent sources).

**Why Thevenin equivalent is important:**

Thevenin theorem allows us to replace a two-terminal network with only two parameters. As such, we only need to solve and/or measure the Thevenin equivalent of a two-port terminal. For example, consider the black box in the figure and the manufacturer spec sheet specifies $v_T = 5 \text{ V}$ and $R_T = 1 \Omega$. Now, if we attach a resistor $R_L$ to this circuit, we can easily calculate $I$ and $V$ without knowing and/or solving for the internals of the black box. For $R_L = 1 \Omega$, we get $I = 2.5 \text{ A}$ and $V = 2.5 \text{ V}$, for $R_L = 4 \Omega$, we get $I = 1 \text{ A}$ and $V = 4 \text{ V}$, etc.

**How to calculate the Thevenin equivalent**

You have seen a detailed discussion of Thevenin/Norton forms in your circuit theory course(s). In summary, the best method is to calculate two of the the following three parameters: (1) Open-circuit voltage, $V_{oc}$ (found by setting $I = 0$), (2) Short-circuit current, $I_{sc}$ (found by shorting the terminals of the two-terminal network, i.e., setting $V = 0$), and (3) Direct calculation of $Z_T$ which is the resistance seen at the terminals with the independent sources.
“killed” (i.e., their strength set equal to zero). Remember, you should NOT “kill” dependent sources. The usual “rule of thumb” is to find $V_{oc}$ and $I_{sc}$ if there is a dependent source in the problem, and to find $V_{oc}$ and $Z_T$ if there is no dependent source in the problem. Then, one can find the Thevenin and Norton parameters from:

$$V_T = V_{oc}; \quad I_n = I_{sc}; \quad I_n Z_T = V_T$$

**Example 1:** Find the Thevenin and Norton Equivalent of the circuit below:

We need to find two of the three parameters $V_{oc}$, $I_{sc}$, and $Z_T$ ($R_T$ here). It is best to find $V_{oc}$ and $Z_T$ for this problem (no dependent source) but all three are calculated for demonstration of the solution technique.

1. $V_{oc}$: Using node-voltage method and noting that since $I = 0$, by KVL, $V_1 = V_{oc}$.

$$\frac{V_1 - 25}{5} - 3 + \frac{V_1}{20} = 0$$

$$4V_1 - 100 - 60 + V_1 = 0$$

$$V_1 = 32 V \quad \Rightarrow \quad V_{oc} = V_T = V_1 = 32V$$

2. $R_T$ (killing the independent sources)

From the circuit, we have

$$R_T = 4 + (5 \parallel 20) = 4 + 4 = 8 \Omega$$

3. $I_{sc}$ To calculate $I_{sc}$ by nodal analysis, note that $I_{sc} = V_1/4$. Then,

$$\frac{V_1 - 25}{5} + \frac{V_1}{4} - 3 + \frac{V_1}{20} = 0$$

$$4V_1 - 100 + 5V_1 - 60 + V_1 = 0$$

$$V_1 = 16 V \quad \Rightarrow \quad I_n = I_{sc} = \frac{V_1}{4} = 4A$$

So, the Thevenin/Norton parameters are: $V_T = 32 V$, $I_n = 4 A$, and $R_T = 8 \Omega$. (note, $v_T = I_n R_T$.)
Thevenin Equivalent of two-terminal networks with controlled sources

Two-terminal networks containing controlled sources also reduce to Thevenin form. However, care should be taken in doing so. As mentioned above, usually the best way to find the Thevenin equivalent of a circuit is to find two of three parameters: $R_T$ (by killing independent sources), $v_T = v_{oc}$ and $i_N = i_{sc}$. The best choice for two-terminal networks containing controlled sources is to find $v_T = v_{oc}$ and $i_N = i_{sc}$ as described in the example below.

**Example:** Find the Thevenin equivalent of this two-terminal network.

Since the circuit has a controlled source, it is preferred to calculate $v_{oc}$ and $i_{sc}$.

**Finding $v_{oc}$**

Since the circuit is simple, we proceed to solve it with KVL and KCL (noting $i = 0$):

\[
\begin{align*}
\text{KCL:} & \quad -i_1 + i + 4i = 0 \quad \rightarrow \quad i_1 = 0 \\
\text{KCL:} & \quad -i_2 - 4i + i_1 = 0 \quad \rightarrow \quad i_2 = 0 \\
\text{KVL:} & \quad -32 + 2 \times 10^3 i_2 + 1.2 \times 10^3 i_1 + v_{oc} = 0 \\
& \quad v_T = v_{oc} = 32 V
\end{align*}
\]

**Finding $i_{sc}$**

Again, using KVL and KCL:

\[
\begin{align*}
\text{KCL:} & \quad -i_1 + i + 4i = 0 \quad \rightarrow \quad i_1 = 5i_{sc} \\
\text{KCL:} & \quad -i_2 - 4i + i_1 = 0 \quad \rightarrow \quad i_2 = i_{sc} \\
\text{KVL:} & \quad -32 + 2 \times 10^3 i_2 + 1.2 \times 10^3 i_1 = 0 \\
& \quad -32 + 2 \times 10^3 i_{sc} + 6 \times 10^3 i_{sc} = 0 \quad \rightarrow \quad i_N = i_{sc} = 4 \times 10^{-3} \text{ A} = 4 \text{ mA}
\end{align*}
\]

Therefore, $v_T = 32 \text{ V}$, $i_N = 4 \text{ mA}$, and $R_T = v_T/i_N = 8 \text{ k\Omega}$.

While finding $v_{oc}$ and $i_{sc}$ is preferred method for most circuits, in some cases, the Thevenin equivalent of the two-terminal network is only a resistor (you will find $v_{oc} = 0$ and $i_{sc} = 0$), or only a voltage source (you will find $v_{oc} \neq 0$ but finding $i_{sc}$ leads to inconsistent or illegal circuits), or only a current source (you will find $i_{sc} \neq 0$ but finding $v_{oc}$ leads to inconsistent or illegal circuits). For these cases, one has to either find $R_T$ directly and/or directly find $i-v$ characteristics of the two-terminal networks as is shown below for the circuit of previous example.
Finding $R_T$

To find $R_T$, we kill all independent sources in the circuit. The resulting circuit cannot be reduced to a simple resistor by series/parallel formulas. This is why finding $v_{oc}$ and $i_{sc}$ is the preferred choices for two-terminal networks containing controlled sources. We can find $R_T$, however, by attaching an ideal voltage source with a known voltage of $v$ and calculate $i$. Since the two-terminal network should be reduced to a resistor ($R_T$), we should get $i = -v/(\text{constant})$ where the constant is $R_T$. (Negative sign comes from active sign convention.)

Since the circuit is simple, we proceed to solve it with KVL and KCL:

$$\text{KCL: } -i_1 + i + 4i = 0 \rightarrow i_1 = 5i$$
$$\text{KCL: } -i_2 - 4i + i_1 = 0 \rightarrow i_2 = i$$
$$\text{KVL: } 0 + 2 \times 10^3 i_2 + 1.2 \times 10^3 i_1 + v = 0$$
$$2 \times 10^3 i + 6 \times 10^3 i + v = 0 \rightarrow i = -\frac{v}{8 \times 10^3}$$

Therefore, $R_T = 8 \times 10^3 \Omega = 8 \text{k}\Omega$.

Note that we could have attached an ideal “current” source with strength of $i$ to the problem, proceeded to calculate $v$, and would have got $v = -8 \times 10^3 i$.

Finding $i-v$ Characteristics Equation:

As mentioned above, in some cases, we have to directly find the $i-v$ characteristics equation in order to find the Thevenin equivalent of a two-terminal network. The procedure is similar to finding $R_T$. Attach an ideal voltage source to the circuit. Assume $v$ is known and proceed to calculate $i$ in terms of $v$. Alternatively, one can attach an ideal current source, assume $i$ is known and find $v$ in terms of $i$. The final expression should look like $v = v_T - iR_T$ and $v_T$ and $R_T$ can be read directly:

Since the circuit is simple, we proceed to solve it with KVL and KCL:

$$\text{KCL: } -i_1 + i + 4i = 0 \rightarrow i_1 = 5i$$
$$\text{KCL: } -i_2 - 4i + i_1 = 0 \rightarrow i_2 = i$$
$$\text{KVL: } -32 + 2 \times 10^3 i_2 + 1.2 \times 10^3 i_1 + v_{oc} = 0$$
$$-32 + 2 \times 10^3 i + 6 \times 10^3 i + v = 0 \rightarrow v = 32 - 8 \times 10^3 i$$

which is the characteristics equation for the two-terminal network and leads to $v_T = 32 \text{ V}$, $R_T = 8 \times 10^3 \Omega = 8 \text{k}\Omega$, and $i_N = v_T/R_T = 4 \text{ mA}$. 

ECE65 Lecture Notes (F. Najmabadi), Winter 2009
How to measure the Thevenin equivalent

Suppose we have given a box with two terminals and want to measure the Thevenin equivalent of the circuit inside the box. In principle, we cannot use the above technique and try to measure $V_{oc}$, $I_{sc}$, and $Z_T$. We cannot turn off the input signal and use an ohm-meter to measure $R_T$. Nor can we short the terminals and measure $I_{sc}$ (there is a good chance that we are going to ruin the circuit if we do that). In principle, we can use a volt-meter (or scope) to measure $V_{oc}$ but care should be taken as it is not known a priori if the internal resistance of the volt-meter (or scope) is large enough to act as an open circuit (there are other complications). There is also the issue of measurement error that one should consider.

To measure $Z_T$ and $V_T$, consider the Thevenin equivalent circuit shown (for simplicity, we are using resistors instead of impedances). From the circuit:

$$\frac{V}{V_T} = \frac{R_L}{R_T + R_L}$$

If we measure $V$ for two different values of $R_L$ (i.e., $R_{L1}$ and $R_{L2}$ with $V_1$ and $V_2$, respectively), the above formula gives:

$$\frac{V_1}{V_T} = \frac{R_{L1}}{R_T + R_{L1}} \quad \text{and} \quad \frac{V_2}{V_T} = \frac{R_{L2}}{R_T + R_{L2}}$$

Dividing the two equations give:

$$\frac{V_1}{V_2} = \frac{R_{L1}}{R_T + R_{L1}} \times \frac{R_T + R_{L2}}{R_{L2}}$$

which can be solved to find $R_T$. One of the above equations for $V_1$ or $V_2$ can then be used to find $V_T$. Typically, we choose $R_{L2}$ to be very large, $R_{L2} \rightarrow \infty$ (e.g., internal resistance of scope). Call $V_2 = V_{oc}$ (open circuit voltage). Then:

$$\frac{V_1}{V_{oc}} = \frac{R_{L1}}{R_T + R_{L1}} \quad \Rightarrow \quad \frac{R_T}{R_{L1}} = \frac{V_{oc}}{V_1} - 1$$

$$V_2 = V_T = V_{oc}$$

So, this measurement will give both gain and output resistance. Note that we should choose $R_{L1}$ such that $V_1$ is sufficiently different from $V_{oc}$ for the measurement to be accurate. Typically, experiment is repeated for several values of $R_{L1}$ until $V_1/V_{oc}$ is between 0.1 to 0.5 (this corresponds to $R_{L1} \sim R_T$).
Exercise: How should one modify the above method if the $R_T$ is actually an impedance, $Z_T$?

An alternate and much more accurate method for cases with resistive $R_T$ is to measure the $IV$ characteristics of the two-terminal network. We can do this by attaching a variable load (a resistance) to the box, vary the load which changes the output voltage and currents, and measure several pair of $I$ and $V$ (here we do not use the value of $R_L$). These data point should lie on the $IV$ line of the two-terminal network. Values of $V_T$, $I_n$, and $R_T$ can be read directly from the graph as shown. This method is specially accurate as one can use a “best-fit” line to the data in order to minimize random measurement errors.

How to find the Thevenin equivalent using PSpice:

You can use the same technique described above for measuring the Thevenin parameters with PSpice. Attach a “variable” load (“Parameter” in PSpice) to the circuit. Ask PSpice to compute output voltage $V$ as a function of load resistance $R_L$. Plot the output current $I$ versus the output voltage $V$ and you will have the $IV$ characteristics of the circuit similar to the figure above (Make sure that you have the current direction correctly!).
1.5 How each sub-circuit sees other elements

The strategy of dividing a circuit into individual components works because of the Thevenin Theorem. Recall that any two-terminal network can be replaced by its Thevenin equivalent. In addition, if a two-terminal network does not include an “independent source” it will be reduced to a single “impedance” (even if it includes dependent sources).

**What Source sees:** The source sees a two-terminal network. This two-terminal network does not contain an independent source. So it can be reduced to a single impedance.

**What Load sees:** The load sees a two-terminal network. This two-terminal network contains an independent source. So it can be reduced to its Thevenin equivalent.

**What each two-port network sees:** Following the logic above, it’s obvious that each two-port network sees a two-terminal network containing an independent source in the input side (can be reduced to a Thevenin form) and a two-terminal network that does not contain an independent source on the output side (so it can be reduced to a single impedance).

The above observations indicate that we do not need to solve a complete circuit. For example, for a particular two-port network, we only need to solve the circuit above (with \( V_s, Z_s, \) and \( Z_L \) as parameters). Then, wherever this two-port network appears in a circuit, we can use these results.
1.6 Transfer Functions

As you can imagine, majority of components in electronic circuits are two-port networks. For example, in a tape recorder, a large number of two-port networks exists between the source (tape head) and the load (speakers). They amplify the signal, filter out the unwanted noise, and process the signal. We study several two-port networks in ECE65. As we noted in the previous page, we can design and analyze these two-port networks using a simple model for the previous stages and a load impedance for later stages of the system as is shown below. If we can model these two-port networks with a finite number of parameters, similar to the Thevenin form for two-terminal networks, we can simplify the analysis of the complete circuit considerably.

Consider the two-port network below. It “communicates” with the outside world (rest of the circuit) through 4 parameters: \( V_i \), \( I_i \), \( V_o \), and \( I_o \). If we solve the two-port network circuit once and find the relationship between these four parameters, we do not need to do that again. While any linear two-port network can be reduced to a combination of four elements (see your circuit theory textbook), it is customary to use the following parameters to describe the behavior of a two-port network.

Voltage transfer function, \( H_v(j\omega) = \frac{V_o}{V_i} \)

Current transfer function, \( H_i(j\omega) = \frac{I_o}{I_i} \)

Equivalent input impedance, \( Z_i(j\omega) = \frac{V_i}{I_i} \)

Equivalent output impedance, \( Z_o(j\omega) = \frac{|V_o|}{|I_o|_{V_s=0}} \)

The equivalent output impedance as defined above is the equivalent Thevenin impedance of a two-terminal network consisting of our 2-port network, \( Z_s \), and \( V_s \). Note that only three of the above four parameters are independent. For example, we can calculate the current transfer function in terms of the other three as will be shown below.

What are \( Z_L \) and \( Z_s \)? Consider a circuit in which our two-port network above is the “nth” two-port network (see figure in the previous page). In this case, the output voltage of “n-1” two-port network is the same as the input voltage of our “nth” two port network: \( V_{o,n-1} = V_{i,n} \) and the output voltage of our “nth” two-port network is the input voltage to the “(n+1)th” two-port network: \( V_{o,n} = V_{i,n+1} \) (with the similar relationship between the currents). The transfer function definitions above indicate that \( Z_L \) is actually the input impedance of “(n+1)th” two-port network (next stage) and \( Z_s \) is the output impedance of
“(n-1)th” two-port network (previous stage). We will see this more clearly as we examine
the various stages in our circuit later.

The case a two-port network with no load attached (usually called “open-loop”) is important (and easier
to solve). The output impedance of this “open-loop”
two-port network is exactly the same as the “terminat-
ed” two-port network (i.e., attached to a load) above. The current gain is zero (since $I_o = 0$).

The voltage transfer function of the “open-loop” two-port network, is:

$$H_{vo}(j\omega) = \frac{V_o}{V_i} = \frac{V_{o,oc}}{Z_{L,\rightarrow\infty}}$$

where $V_{o,oc}$ is the “open-loop” (or open circuit) output voltage.

To see the relationship between the “open-loop” and “termi-
nated” transfer functions of a two-port network, let us replace the
two-port network, $V_s$ and $Z_s$ (parts in the dashed box in the fig-
ure above) with its Thevenin equivalent. The Thevenin voltage source, $V_T$, is the “open-loop” voltage across the output terminals:

$$V_T = V_{o,oc} = H_{vo}(j\omega)V_i$$

and the Thevenin impedance is by definition the output impedance of the two-port network.

We can now derive a “terminated” two-port network transfer functions in terms of its “open-
loop” transfer functions by using the Thevenin equivalent circuit:

$$V_o = \frac{Z_L}{Z_L + Z_o} H_{vo}(j\omega)V_i$$

$$H_v(j\omega) = \frac{V_o}{V_i} = \frac{Z_L}{Z_L + Z_o} H_{vo}(j\omega)$$

The output impedance of the terminated two-port network is exactly the same as the “open-
loop” version as mentioned before. The current gain of the terminated two-port network can
be found from:

$$I_o = \frac{1}{Z_L + Z_o} H_{vo}(j\omega)V_i \quad \text{and} \quad I_i = \frac{V_i}{Z_i}$$

$$H_i(j\omega) = \frac{I_o}{I_i} = \frac{Z_i}{Z_L + Z_o} H_{vo}(j\omega)$$
In general, the input impedance of the terminated two-port network is different from that of the “open-loop” version because when a load is attached to a two-port network, the input current will change. For most cases (and for well-designed circuits), however, this effect is small and the two input impedances are approximately the same.

Lastly, the expression for $H_v(j\omega)$ shows that if $Z_L \gg Z_o$, the terminated transfer function becomes equal to the open-loop one (one can show the input impedances of open-loop and terminated networks also become the same). This is an important concept and consideration which we will examine more below.

**Interaction between Components**

Let us consider what happens when many two-port networks are attached to each other. We start with the first two-port networks of the general circuit figure of page 11 as is shown with the “source” replaced with its Thevenin equivalent. We see that the output voltage of first two-terminal network is the same as the input voltage of the second one, $V_{o,1} = V_{i,2}$.

Also, since by definition $Z_{i,2} = V_{i,2}/I_{i,2}$, the two-port network no. 1 sees the input impedance of the 2nd two-port network as its load as is shown.

The output voltage, $V_{o,1}$ can be written in terms of the open-loop transfer function of first two-port network:

$$V_{o,1} = H_v V_{i,1} = \frac{Z_{i,2}}{Z_{i,2} + Z_{o,1}} H_{vo,1} V_{i,1}$$

We can now proceed to find the output voltage of the 2nd two-terminal network by noting that its load is the input-resistance of the third two-port network. Thus,

$$V_{o,2} = H_v V_{i,2} = \frac{Z_{i,3}}{Z_{i,3} + Z_{o,2}} H_{vo,2} V_{i,2}$$

Since the input voltage to the 2nd two-port network is $V_{i,2} = V_{o,1}$, we get:

$$V_{o,2} = \frac{Z_{i,3}}{Z_{i,3} + Z_{o,2}} \times \frac{Z_{i,2}}{Z_{i,2} + Z_{o,1}} \times H_{vo,2} \times H_{vo,1} V_{i,1}$$
So, the first two two-port networks act like one two-port network with the following transfer characteristics:

\[ H(j\omega) = \frac{V_{o,2}}{V_{i,1}} = \frac{Z_{i,3}}{Z_{i,3} + Z_{o,2}} \times \frac{Z_{i,2}}{Z_{i,2} + Z_{o,1}} \times H_{vo,2} \times H_{vo,1} \]

The input impedance would be the input impedance of the first two-port network and the output impedance would be the output impedance of the second two-port network.

Similarly we can find the transfer function of “N” two-port networks placed behind each other. The input impedance would be the input of the first two-port network, the output impedance would be the output impedance of the “Nth” (last) two-port network and the voltage transfer function will be proportional to \( H_{vo,1}H_{vo,2} \cdots H_{vo,N} V_i \) multiplied by fractions that includes the input and output resistances of various stages.

**Voltage and Power Transfer**

It is obvious from the above that when a two-port network is placed in a circuit, the output impedance of the previous stage \( Z_s = Z_{o,n-1} \) and the input impedance of the next stage \( Z_L = Z_{i,n+1} \) affect the two-port network transfer functions.

One is interested to find if there is a way to ensure good coupling and maximum signal transfer between connecting two-port networks. Consider the connection between two two-port network as is shown below.

As we are interested in the interaction between the “nth” and “(n+1)th” two-port networks, we replace the “nth” two-port network and all of the circuit to its left with its Thevenin equivalent with \( Z_s = Z_{o,n} \) and \( V_s = H_{vo,n} V_{i,n} \) (see page 13). Similarly the “(n+1)th” two-port network and all of the circuitry to its right can be replaced by \( Z_L = Z_{i,n+1} \) (see page 13)

Good coupling between components typically means largest \( I_L, V_L \) or power, \( P_L = V_L I_L \). Unfortunately, these three parameters do not maximize simultaneously.

\[
I_L = \frac{V_s}{Z_s + Z_L} \quad V_L = \frac{Z_L}{Z_s + Z_L} V_s \quad P_L = V_L I_L = \frac{Z_L}{(Z_s + Z_L)^2} V_s^2
\]
Values of $I_L$, $V_L$, and $P_L$ are plotted in the figure assuming $V_s$ and $Z_s$ are fixed. We can see that best current coupling (maximum $I_L$) when $Z_L = 0$ (or effectively, $Z_L/Z_s \ll 1$) and the best voltage coupling (maximum $V_L$) when $Z_L \to \infty$ (or effectively, $Z_L/Z_s \gg 1$). The best power coupling (maximum $P_L$) is somewhere in between when $Z_L = Z_s^*$ ($R_L = R_s$ and $X_L = X_s$).

Maximum voltage transfer: $\frac{Z_L}{Z_s} \gg 1 \rightarrow V_L|_{max} = V_s$

Maximum current transfer: $\frac{Z_L}{Z_s} \ll 1 \rightarrow I_L|_{max} = \frac{V_s}{Z_s}$

Maximum power transfer: $Z_L = Z_s^* \rightarrow P_L|_{max} = \frac{V^2}{4Z_s}$

Maximum power transfer is not usually a criteria for coupling components (except the last stage of coupling to the load). In most cases, we are interested in good voltage coupling to keep power dissipation in the circuit small. For example, consider a CD player. The source produces a low-voltage signal proportional to the information on the CD which needs to be amplified, translated into sound frequencies, amplified further, filtered, etc. and then fed to a speaker. In order to keep the circuit small and cheap, we amplify the signal and do the signal processing with signals of substantial voltage but low current. This keep the power dissipation in each stage small. Only in the last stage (power amplifier) the signal current is increased to drive the load (speakers).

The criteria for best voltage coupling is $Z_L \gg Z_s$ ($V_L \approx V_s$). If we are modeling the interaction between two two-port networks, $Z_s$ represent the output impedance of the previous stage, $Z_L$ represents the input resistance of the next stage. Therefore, best voltage coupling condition translates into ensuring that output impedance of previous stage is much smaller than input impedance of the next stage.:

$Z_{o,n} \ll Z_{i,n+1}$

And, a useful goal for designing two-port networks is to ensure that input impedance is large and output impedance is small.

With this constraint, we can readily see that each two-port network effectively sees an “infinite” load and the output voltage of our nth two-port network simplifies to

$V_{o,n} = H_{v_{o,1}}H_{v_{o,2}}\cdots H_{v_{o,n}} \times V_s$
Impedance Matching at High Frequencies: At high frequencies, the distance that an electric signal travels in one period may become comparable to dimension of the circuit or the cables that connect the circuit to other devices. Light (or electric signals) travel 3 m in 10 ns (period of 100 MHz signal). You will see in ECE107 that electric signals can be reflected from any element. This reflection leads to the main signal being followed with signals with similar shape but smaller amplitude that appear with a time delay from the main signal (like light bouncing between two mirrors). This is called “ringing” in the circuit. Ringing can be eliminated if the output impedance of a circuit is exactly the same as the input impedance of the next stage (called “impedance matching”). Obviously having a “clean” signal is more important that the best voltage coupling so impedance matching criteria ($Z_{o,n} = Z_{i,n+1}$) is usually used.

As the circuit frequency is increased, problem with “ringing” will first appear in the cables that connect different high-frequency devices to each other as they are usually the longest path between elements. Most devices use coaxial cable (with a BNC connector) for connection. These coaxial cables have an impedance of 50 Ω at high frequencies. Thus, most devices designed to operate at high frequency and use a BNC coaxial cable as connecting wire have an output impedance of 50 Ω and an input impedance of 50 Ω (to match the BNC cable).

In ECE65, we work only with low-frequency circuits (smaller than a few MHz) so we do not need to worry about impedance matching and we will use maximum voltage coupling criteria above.

1.7 Component Analysis Procedure in ECE65

In this course, we examine many two-port networks, calculate their parameters (transfer functions and input and output impedances), and experiment with them in the Lab.

We showed that in general we need to solve a circuit as is shown. Examination of the definitions of parameters of the two-port network shows that only $Z_o$ depends on $Z_s$. Furthermore, if we follow the good practice of designing circuits with low output impedance and high input impedance, one can easily show that $Z_o$ will become independent of $Z_s$. So, for the rest of this course, $Z_s$ is ignored (then, $V_s = V_i$). We also explored the connection between open-loop and terminated transfer functions of two-port networks. As such, we will first examine the open-loop voltage transfer function for each two-port network. The impact of the load is discussed afterward.
1.7.1 How to measure $H_v, Z_i, Z_o$

Measuring $H_v$ is straightforward. Apply a sinusoidal input signal with amplitude $V_i$ to the input, measure the amplitude of the input and output signals with a very large load (scope input resistance). Then $|H_v(\omega)| = |V_o|/|V_i|$ and $\angle H_v(\omega) = \angle V_o - \angle V_i$. We need to repeat this measurement for many frequencies.

In the measurements above (and those of $Z_i$ and $Z_o$), it is essential to ensure that the circuit behaves “linearly” (i.e., if the input signal is a sine wave, the output signal should also be a sine wave). Circuit typically operate linearly over a range of parameters (e.g., frequencies, voltages, currents). Definitions of $H_v$, $Z_i$, and $Z_o$ are meaningful only for “linear” circuits.

Measuring input resistance:

Measuring input and output resistances (or impedances) is not trivial. Part of the difficulty is due to the fact that an Am-meter is limited to 60 Hz (good one can measure up to a few kHz). Scopes which operate a high frequencies, however, only measure voltages. In order to measure a current with scope, we need to find the voltage across a resistor whose value is accurately known (and use Ohm’s Law!). Care should be taken as addition of such a resistor to the circuit may modify its behavior. The general technique for measuring input and output resistance of any two-port network is given below. For simplicity, we focus on the cases where the input and output impedances are resistors. The procedure can easily be generalized to impedances.

Add a resistance $R_1$ (measured accurately) in series with the input. Apply a sine wave to the input. Measure voltages $V_i$ and $V_A$. Then:

$$\frac{R_i}{R_1 + R_i} = \frac{V_i}{V_A}$$

$$\frac{R_1}{R_i} = \frac{V_A}{V_i} - 1$$

Note that $R_1$ should be chosen such that a) $V_i$ is not “too” small to be measured accurately (large $R_1$ compared to $R_i$), and b) $V_A$ and $V_i$ are not “too” close (small $R_1$ compared to $R_i$).

To see the second point, assume we measure $V_A$ and $V_i$ to be 1 and 0.98 V respectively. Also, $R_1 = 1$ kΩ. If we use these values in the above formula, we will find $R_i = 49$ kΩ.

Let us now consider the error in our measurements. The measurements are performed with a scope with a relative accuracy of 2%. In this case, the “real” value of $V_A$ is in the range
of $1 \pm 0.02 \times 1$ V or between 0.98 to 1.02 V. Similarly $V_i$ would be in the range of 0.96 to 1 V. The term $V_A - V_i$ which appears in the formula for $R_i/R_i$ above would then be in the range of $-0.02$ to 0.06 V. Since negative values of $V_A - V_i$ is unphysical, we should have $V_A - V_i \leq 0.06$. The above formula for $R_i$ then gives $R_i \geq 16$ kΩ. It can be seen clearly that the first calculation leading to $R_i = 49$ kΩ is incorrect. $R_i$ can be as large as several MΩ or as small as 16 kΩ. This problem always occurs when we are interested in the difference between two measurements that are close to each other.

To avoid the above problem, experiment usually is repeated for different values of $R_1$ until $V_i/V_A$ is between 0.2 to 0.8 ($V_i/V_A = 0.5$ corresponds to $R_1 = R_i$).

**Measuring output resistance:** To measure $R_o$, consider the two-port network model of page 13 with $Z_L$ replaced with $R_L$.

From the circuit:

$$\frac{V_o}{H_v V_i} = \frac{R_L}{R_o + R_L}$$

If we measure $V_o$ for two different values of $R_L$ (i.e., $R_{L1}$ and $R_{L2}$ with $V_{o1}$ and $V_{o2}$, respectively), the above formula gives:

$$\frac{V_{o1}}{H_v V_i} = \frac{R_{L1}}{R_o + R_{L1}}$$
$$\frac{V_{o2}}{H_v V_i} = \frac{R_{L2}}{R_o + R_{L2}}$$

If the two measurements are done for the same value $V_i$, dividing the two equations gives:

$$\frac{V_{o1}}{V_{o2}} = \frac{R_{L1}}{R_o + R_{L1}} \times \frac{R_o + R_{L2}}{R_{L2}}$$

which can be solved to find $R_o$. Typically, we choose $R_{L2}$ to be very large, $R_{L2} \to \infty$ (e.g., internal resistance of scope). Call $V_{o2} = H_v V_i = V_{oc}$ (open circuit voltage). Then:

$$\frac{V_{o1}}{V_{oc}} = \frac{R_{L1}}{R_o + R_{L1}}$$
$$\frac{R_o}{R_{L1}} = \frac{V_{oc}}{V_{o1}} - 1$$

Similar to measuring the input resistance, we should choose $R_{L1}$ such that $V_{o1}$ is sufficiently different from $V_{oc}$ for the measurement to be accurate. As in the case of measuring input resistance, several values of $R_{L1}$ are used until $V_{o1}/V_{oc}$ is typically between 0.2 to 0.8 ($V_{o1}/V_{oc} = 0.5$ corresponds to $R_{L1} = R_o$).
In some cases, we cannot reduce $R_{L1}$ to a low enough level such that $V_{oc}/V_{o1}$ is sufficiently different from 1. This is due to the maximum output current limitation (we will revisit this for OpAmps). In such cases, we cannot find the value of the output resistance. However, we can still find a “bound” for $R_o$. Suppose $R_{L1}$ is the “smallest” resistance (and circuit still behaves linearly) that we can use and at this value of $R_{L1}$ and to accuracy of our scope ($\epsilon = 2\%$), $V_{oc} \approx V_o$.

If the relative measurement error is $\epsilon$, then:

$$V_{oc|measured} (1 - \epsilon) \leq V_{oc|real} \leq V_{oc|measured} (1 + \epsilon)$$

$$V_{o1|measured} (1 - \epsilon) \leq V_{o1|real} \leq V_{o1|measured} (1 + \epsilon)$$

$$1 - 2\epsilon \leq \frac{V_{oc}}{V_{o1}}|_{real} \leq 1 + 2\epsilon$$

$$\frac{R_{o1}}{R_{L1}} \leq 2\epsilon \quad \rightarrow \quad R_o \leq 2\epsilon R_{L1}$$

So, for example, if the accuracy of the scope is 2% and the minimum $R_{L1}$ we could use was 1 kΩ, the bound for $R_o \leq 2 \times 0.02 \times 1000 = 40 \, \Omega$.

### 1.8 Voltage divider as an example of a two-port network

Let’s consider what we have discussed in the context of a simple circuit:

$$I_i = \frac{V_s}{R_s + R_1 + R_2 \parallel R_L}$$

$$I_o = -\frac{V_o}{R_L}$$

$$V_o = (R_2 \parallel R_L) I_i = \frac{R_2 \parallel R_L}{R_s + R_1 + R_2 \parallel R_L} V_s$$

$$V_i = (R_1 + R_2 \parallel R_L) I_i = \frac{R_1 + R_2 \parallel R_L}{R_s + R_1 + R_2 \parallel R_L} V_s$$

From the above we can now calculate the parameters of our two-port network:

$$H_o = \frac{V_o}{V_i} = \frac{R_2 \parallel R_L}{R_1 + R_2 \parallel R_L} \quad H_i = -\frac{I_o}{I_i} = \frac{R_2 \parallel R_L}{R_L}$$

$$Z_i = \frac{V_i}{I_i} = R_1 + R_2 \parallel R_L \quad Z_o = \frac{V_o|_{V_s=0}}{I_o} = (R_s + R_1) \parallel R_2$$
Note that the transfer functions are independent of the value of input signal strength, \( V_i \) (because the circuit is linear) and the parameters of the two-port network “depend” on values of \( R_s \) and \( R_L \). For this case, the transfer functions are frequency independent (no capacitor or inductor in the circuit).

The open-loop transfer functions can be found from the above formulas by letting \( R_L \to \infty \):

\[
H_v|_{\text{max}} = H_{vo} = \frac{R_2}{R_1 + R_2}
\]

Note that the analysis of best voltage coupling on page 16 indicates that \( H_v \) is maximum when \( R_L \to \infty \):

**Note:** A large number of circuits include resistor attached between the output terminals (like \( R_2 \) in the voltage divider above). It is essential to note that these resistors are NOT necessarily the output resistance of the circuit (compare value of \( Z_o \) above with \( R_2 \)).

### 1.9 Mathematics versus Engineering

You should have learned by now that one cannot achieve “mathematical” accuracy in practical systems. Firstly, our instruments have a finite accuracy in measuring values. When a number (or measurement), \( A \), has a tolerance of \( \epsilon \), it means that its value is between \( A(1 \pm \epsilon) = A \pm \epsilon A \). This means that we cannot differentiate between any number in the range \( A - \epsilon A \) to \( A + \epsilon A \). We would say that all numbers in this range are “approximately equal” to each other:

\[
B \approx A \quad \iff \quad A - \epsilon A \leq B \leq A + \epsilon A
\]

and we can use \( B \) and \( A \) interchangeably as we cannot distinguish between them.

As an example, the scopes in ECE65 lab are accurate within 2%. So, if the scope (\( \epsilon = 0.02 \)) reads a value of 1.352 V, the “real” value is anywhere between 1.352 ± 0.02 × 1.352 or in the range of 1.325 to 1.379. In this context any number between 1.325 and 1.379 is approximately equal to 1.352 as we CANNOT differentiate among them by our measurements: 1.325 ≈ 1.352 and 1.379 ≈ 1.325.

**Corollary:** In the example above, the 4th significant digits in 1.352 is totally meaningless (see the range of numbers we cannot distinguish). It is a poor engineering practice to even report this 4th significant digit! (Still some ECE65 students report their calculations to 8th significant digits, directly writing the number from their calculators!). Similarly, it is
poor engineering practice to report numbers in whole fractions (e.g., 4/3). No measuring instrument measure any property in whole numbers!

Secondly, each element/component/system is manufactured to a certain tolerance – the smaller the tolerance, the more expensive is to build that component. For example, resistors we will use in the Lab have a tolerance of 5%. This means that a 1 kΩ has a value of 1,000 ± 5% = 1,000 ± 50 Ω or somewhere between 950 and 1,050 Ω.

A corollary of this concept is that if you designed a circuit and found that you need a 1,010 Ω resistor, you CANNOT put a 1 kΩ and a 10 Ω resistor (with 5% tolerance) in series. The resultant combination would have a value between 959.5 and 1,060.5 Ω which is no better than a 5% 1 kΩ resistor. If you need to have a 1,010 Ω resistor (i.e., more precision), you should use 1% resistors (which are more expensive).

Concept of infinity and zero (or big and small) also are meaningless in abstract. For example, we saw that the transfer functions of a terminated two-port network is similar to the corresponding open-loop ones (derived assuming $Z_L \to \infty$) when $Z_L \gg Z_o$. So, for a two-port network, “infinite” load resistance means $Z_L \gg Z_o$. We cannot state what the value of an “infinite” resistance is. In the above example, if $Z_o = 1$ Ω, a 100 Ω load resistance would be infinite, while if $Z_o = 1000$ Ω, a 100 Ω load resistance would actually be small. So, concepts of large and small require a frame of reference, i.e., big or small compared to what, and should be stated as “much smaller” or “much bigger” than ···.

Notions of much smaller ($\ll$) and much greater ($\gg$) are meaningful only in term of a given or needed “relative” tolerance, $\epsilon$. Consider quantity $B = A + a$. We use the concept of much smaller, $a \ll A$, to write $B \approx A$. From the above definition of approximate, we should have (assuming that $a$ and $A$ are positive):

$$B \approx A \quad \rightarrow \quad A - \epsilon A \leq B \leq A + \epsilon A$$

$$A - \epsilon A \leq A + a \leq A + \epsilon A \quad \rightarrow \quad a \leq \epsilon A$$

$$a \ll A \quad \Rightarrow \quad a \leq \epsilon A$$

In the above example with scope, we note $1.352 = 1.35 + 0.002$. Since $0.002 = 0.0015 \times 1.35 < 0.01 \times 1.35$, we can ignore the 4th significant digits “2”

**Exercise:** Show that with a tolerance of $\epsilon$, $A \gg a$ means $A \geq (1/\epsilon)a$.

For most day-to-day use, a tolerance of 5% to 10% is more than sufficient. As a general rule, we will use a tolerance of 10% in the analysis in ECE65 unless otherwise stated.