Linear Amplifiers and OpAmps

Amplifiers are two-port networks in which the output voltage or current is directly proportional to either input voltage or current. Four different kinds of amplifiers exist:

- **Voltage amplifier:** \( A_v = \frac{V_o}{V_i} = \text{constant} \)
- **Current amplifier:** \( A_i = \frac{I_o}{I_i} = \text{constant} \)
- **Transconductance amplifier:** \( G_m = \frac{I_o}{V_i} = \text{constant} \)
- **Transresistance amplifier:** \( R_m = \frac{V_o}{I_i} = \text{constant} \)

Our focus in this course is on voltage amplifiers (we also see a transconductance amplifier).

Voltage amplifiers can be accurately modeled with three circuit elements as shown below. These circuit elements are related to transfer functions of two-port networks discussed before. \( R_i \) and \( R_o \) are, respectively, input and output resistances. Gain, \( A_0 \), is the value of \( H(j\omega) \) of the un-terminated circuit. (For voltage amplifiers the convention is to use \( A_v \) instead of \( H_v(j\omega) \)). A good voltage amplifier has a large input resistance, \( R_i \), and a small output resistance, \( R_o \). An ideal voltage amplifier has, \( R_i \to \infty \) and \( R_o \to 0 \).

Practical Amplifiers

Real voltage amplifiers differ from the ideal amplifiers. Not only, the input resistance is not infinite and the output resistance is not zero, but the amplifier works properly only in certain conditions. One should always be aware of the range where the circuit acts as a linear (ideal) amplifier, i.e., the output is proportional to the input with the ratio of \( A_v = \frac{V_o}{V_i} = \text{constant} \) (exactly the same waveform).

**Amplifier Saturation:** Amplifiers do not create power. Rather, they act as a “valve” adjusting the power flow from the power supply into the load according to the input signal. As such, the output voltage amplifier cannot exceed the power supply voltage (it is usually lower because of voltage drop across some active elements). The fact that the output voltage of a practical amplifier cannot exceed certain threshold value is called saturation. A voltage amplifier behaves linearly, i.e., \( V_o/V_i = A_v = \text{constant} \) as long as the output voltage remains
below the “saturation” voltage,

\[-V_{\text{sat}} < V_o < V_{\text{sat}}\]

Note that the saturation voltage, in general, is not symmetric, i.e., \(-V_{\text{sat},1} < V_o < V_{\text{sat},2}\).

For an amplifier with a given gain, \(A_v\), the above range of \(V_o\) translate into a certain range for \(V_i\)

\[-V_{\text{sat}} < V_o < V_{\text{sat}}\]
\[-V_{\text{sat}} < A_v V_i < V_{\text{sat}}\]
\[-\frac{V_{\text{sat}}}{A_v} < V_i < \frac{-V_{\text{sat}}}{A_v}\]

i.e., any amplifier will enter its saturation region if \(V_i\) is raised above certain limit. The figure shows how the amplifier output clips when amplifier is not in the linear region.

**Amplifier Bandwidth:** Typically, amplifiers work in a certain range of frequencies. Their gain, \(A_v = V_o/V_i\) drops outside this range. The voltage transfer function (gain) of an amplifier is plotted similar to that of filters (Bode plots). It looks similar to those of a band-pass filter or a low-pass filter. Cut-off frequencies and bandwidth of the amplifier is defined similar to those of filters (3 dB drop from the maximum value). In terms of frequency response, voltage amplifiers are divided into two categories. (1) AC amplifiers which only amplify AC signals. Their Bode plots look like a band-pass filter. (2) DC amplifiers which amplify both DC signals and AC signals up to a certain frequency. Their Bode plots look like a low-pass filter. For this class amplifiers, the bandwidth is equal to the cut-off frequency (lower cut-off frequency is set to zero!).

**Rise Time:** In an ideal amplifier, if the input voltage is a unit step function, the output voltage will also be a unit step function as shown. A practical amplifier cannot change its output instantaneously if the input changes suddenly. It takes some time (a short but finite time) for the amplifier output voltage to reach its nominal level. The maximum rate of change in the output voltage is called the rise time.
**Maximum Output Current:** Voltage amplifiers are designed to amplify the voltage and not the current. However, nothing prevents a user to attach a small load to the output, drawing a large current. As the load resistance is decreased, the output current of amplifier is increased. At the certain output current, the output signal of the amplifier does not resemble the input signal. This is due to “maximum output current limitation.”

There are other limitations of a practical amplifier such as signal-to-noise ratio, nonlinear distortion, etc. which are out of the scope of this course.

**How to measure** $A_v, R_i, R_o$

Measuring $A_v$ is straightforward. Apply a sinusoidal input signal with amplitude $V_i$ to the input, measure the amplitude of the input and output signals with a very large load (scope input resistance). Make sure that the amplifier is not in saturation and signal is not clipped), and compute $A_v = V_o/V_i$. Note that $A_v$ is in general depends on frequency. There may also been some phase difference between $V_i$ and $V_o$ similar to filters.

Measuring input and output resistances (or impedances) is not trivial. Part of the difficulty is due to the fact that an Am-meter is limited to 60 Hz (good one can measure up to a few kHz) and we should use scopes that only measure voltages. In order to measure a current with scope, we need to find the voltage across a resistor whose value is accurately known. Care should be taken as addition of such a resistor to the circuit may modify its behavior. The general technique for measuring input and output resistance of any two-port network is given below:

**Measuring input resistance:**

Add a resistance $R_1$ (measured accurately) in series with the input. Measure voltages $V_i$ and $V_A$. Then:

\[
\frac{R_i}{R_1 + R_i} = \frac{V_i}{V_A} \\
R_i = \frac{V_A}{V_i} - 1
\]

Note that $R_1$ should be chosen such that a) $V_i$ is not “too” small to be measured accurately (large $R_1$ compared to $R_i$), and b) $V_A$ and $V_i$ are not “too” close (small $R_1$ compared to $R_i$). Typically, experiment is repeated for different values of $R_1$ until $V_i/V_A$ is between 0.1 to 0.5 (this corresponds to $R_1 \sim R_i$)
Measuring output resistance:

To measure $R_o$, consider the amplifier model of page 27 with $Z_L$ replaced with $R_L$. From the circuit:

$$\frac{V_o}{A_0V_i} = \frac{R_L}{R_o + R_L}$$

If we measure $V_o$ for two different values of $R_L$ (i.e., $R_{L1}$ and $R_{L2}$ with $V_{o1}$ and $V_{o2}$, respectively), the above formula gives:

$$\frac{V_{o1}}{A_0V_i} = \frac{R_{L1}}{R_o + R_{L1}}$$
$$\frac{V_{o2}}{A_0V_i} = \frac{R_{L2}}{R_o + R_{L2}}$$

If the two measurements are done for the same value $V_i$, dividing the two equations give:

$$\frac{V_{o1}}{V_{o2}} = \frac{R_{L1}}{R_o + R_{L1}} \times \frac{R_o + R_{L2}}{R_{L2}}$$

which can be solved to find $R_o$. Typically, we choose $R_{L2}$ to be very large, $R_{L2} \rightarrow \infty$ (e.g., internal resistance of scope). Call $V_{o2} = V_{oc}$ (open circuit voltage). Then:

$$\frac{V_{o1}}{V_{oc}} = \frac{R_{L1}}{R_o + R_{L1}}$$
$$\frac{R_o}{R_{L1}} = \frac{V_{oc}}{V_{o1}} - 1$$

We also note:

$$V_{o2} = AV_i = V_{oc} \quad \rightarrow \quad A = \frac{V_{oc}}{V_i}$$

So, this measurement will give both gain and output resistance. Note that we should choose $R_{L1}$ such that $V_{o1}$ is sufficiently different from $V_{oc}$ for the measurement to be accurate. As in the case of measuring input resistance, several values of $R_{L1}$ are used until $V_{o1}/V_{oc}$ is typically between 0.1 to 0.5 (this corresponds to $R_{L1} \sim R_o$)

**Exercise:** Show that the technique above is “mathematically” the same as that described for measuring Thevenin resistance of a circuit described in page 5. Explain why we are not using that technique here.
In some cases, we cannot reduce $R_{L1}$ to a low enough level such that $V_{oc}/V_{o1}$ is sufficiently different from 1. This is due to the maximum output current limitation (we will revisit this for OpAmps). In such cases, we can still find a “bound” for $R_o$. Suppose $R_{L1}$ is the smallest resistance that we can use and at this value of $R_{L1}$, $V_{oc} \approx V_o$. For lower values the amplifier is not linear, i.e., the output signal is different from input.

If the relative measurement error is $\epsilon$, then:

\[
\frac{V_{o1}}{V_{o1}}_{\text{real}} \leq \frac{V_{o1}}{V_{o1}}_{\text{measured}} (1 \pm \epsilon) \\
\frac{V_{oc}}{V_{o1}}_{\text{real}} \leq \frac{V_{oc}}{V_{o1}}_{\text{measured}} (1 \pm \epsilon) \\
\frac{R_{o1}}{R_o} \leq 2\epsilon \quad \rightarrow \quad R_o \leq 2\epsilon R_L
\]

So, for example, if the accuracy of the scope is 3% and the minimum $R_L$ we could use was 1 k\(^\Omega\), the bound for $R_o \leq 2 \times 0.03 \times 1000 = 60 \ \Omega$.

**Feedback**

Not only a good amplifier should have sufficient gain, its performance should be insensitive to environmental and manufacturing conditions, should have a large $R_i$, a small $R_o$, a sufficiently large bandwidth, etc. It is easy to make an amplifier with a very large gain. A typical transistor circuit can easily have a gain of 100 or more. A three-stage transistor amplifier can easily get gains of $10^6$. Other characteristics of a good amplifier is hard to achieve. For example, the $\beta$ of a BJT changes with operating temperature making the gain of the three-stage amplifier vary widely. The system can be made to be insensitive to environmental and manufacturing conditions by the use of feedback. Feedback also helps in other regards.

**Principle of feedback:** The input to the circuit is modified by “feeding” a signal proportional to the output value “back” to the input. There are two types of feedback (remember the example of a car in the freeway discussed in the class):

1. **Negative feedback:** As the output is increased, the input signal is decreased and *vice versa*. Negative feedback stabilizes the output to the desired level. Linear system employs negative feedback.

2. **Positive feedback:** As the output is increased, the input signal is increased and *vice versa*. Positive feedback leads to instability. (But, it has its uses!)

We will explore the concept of feedback in the context of operational amplifiers.
OpAmps as linear amplifiers

Operational amplifiers (OpAmps) are general purpose voltage amplifiers employed in a variety of circuits. OpAmps are “DC” amplifiers with a very large gain, $A_0$ ($10^5$ to $10^6$), high input impedance ($> 1 – 10 \text{ M}\Omega$), and low output resistance ($< 100 \text{ \Omega}$). They are constructed as a “difference” amplifier, i.e., the output signal is proportional to the difference between the two input signals.

$$V_o = A_0V_d = A_0(V_p - V_n)$$

$V_s$ and $-V_s$ are power supply attachments. They set the saturation voltages for the OpAmp circuit (within 0.2 V). Power supply ground should also be connected to the OpAmp ground. + and – terminals of the OpAmp are called, respectively, non-inverting and inverting terminals.

OpAmp Models

Because $R_i$ is very large and $R_o$ is very small, ideal model of the OpAmp assumes $R_i \to \infty$ and $R_o \to 0$. Ideal model is usually a very good model for OpAmp circuits. Very large input resistance also means that the input current into an OpAmp is very small:

**First Golden Rule of OpAmps:** $I_p \approx I_n \approx 0$ (Also called “Virtual Open Principle”)

An important feature of OpAmp is that because the gain is very high, the OpAmp will be in the saturation region without negative feedback. For example, take an OpAmp with a gain of $10^5$ and $V_{sat} = 15 \text{ V}$. Then, for OpAmp to be in linear region, $V_i \leq 15 \times 10^{-5} = 150 \mu\text{V}$ (a very small value). OpAmps are never used by themselves. They are always part of a circuit which employ either negative feedback (e.g., linear amplifiers, active filters) or positive feedback (e.g., comparators). Examples below shows several OpAmp circuits with negative feedback.
The first step in solving OpAmp circuits is to replace the OpAmp with its circuit model (ideal model is usually very good).

\[ V_p = 0, \quad V_o = A_0 V_d = A_0 (V_p - V_n) = -A_0 V_n \]

Using node-voltage method and noting \( I_n \approx 0 \):

\[ \frac{V_n - V_i}{R_1} + \frac{V_n - V_o}{R_2} = 0 \]

Substituting for \( V_n = -V_o/A_0 \) and multiplying the equation by \( R_2 \), we have:

\[ - \frac{R_2}{A_0 R_1} V_o - \frac{R_2}{R_1} V_i - \frac{V_o}{A_0} - V_o = 0 \quad \rightarrow \quad V_o \left[ 1 + \frac{1}{A_0} + \frac{R_2}{A_0 R_1} \right] = \frac{R_2}{R_1} V_i \]

\[ \frac{V_o}{V_i} = - \frac{R_2}{R_1} \left[ 1 + \frac{1}{A_0} + \frac{R_2}{A_0 R_1} \right] \]

Since OpAmp gain is very large, \( 1/A_0 \ll 1 \). Also if \( R_2 \) and \( R_1 \) are chosen such that their ratio is not very large, \( R_2/R_1 \ll A_0 \) or \( R_2/(A_0 R_1) \ll 1 \), then the voltage transfer function of the OpAmp is

\[ \frac{V_o}{V_i} = - \frac{R_2}{R_1} \]

The circuit is called an inverting amplifier because the voltage transfer function is “negative.” (A “negative” sinusoidal function looks inverted.) The negative sign means that there is 180° phase shift between input and output signals.

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Note that the voltage transfer function is “independent” of the OpAmp gain, \( A_0 \), and is only set by the values of the resistors \( R_1 \) and \( R_2 \). While \( A_0 \) is quite sensitive to environmental and manufacturing conditions (can vary by a factor of 10 to 100), the resistor values are quite insensitive and, thus, the gain of the system is quite stable. This stability is achieved by negative feedback, the output of the OpAmp is connected via \( R_2 \) to the inverting terminal of OpAmp. If \( V_o \) increases, this resistor forces \( V_n \) to increase, reducing \( V_d = V_p - V_n \) and \( V_o = A_0 V_d \), and stabilizes the OpAmp output.

An important feature of OpAmp circuits with negative feedback is that because the OpAmp is NOT saturated, \( V_d = V_o/A_0 \) is very small (because \( A_0 \) is very large). As a result,

\[
\text{Negative Feedback} \quad \rightarrow \quad V_d \approx 0 \quad \rightarrow \quad V_n \approx V_p
\]

**Second Golden Rule of OpAmps:** For OpAmps circuits with negative feedback, the OpAmp adjusts its output voltage such that \( V_d \approx 0 \) or \( V_n \approx V_p \) (also called “Virtual Short Principle”). This rule is derived by assuming \( A \rightarrow \infty \). Thus, \( V_o \) cannot be found from \( V_o = A_0 v_d = \infty \times 0 = \text{indefinite value} \). The virtual short principle replace \( V_o = A_0 v_d \) expression with \( V_d \approx 0 \).

The above rule simplifies solution to OpAmp circuits dramatically. For example, for the inverting amplifier circuit above, we will have:

\[
\begin{align*}
\text{Negative Feedback} & \quad \rightarrow \quad V_n \approx V_p \approx 0 \\
\frac{V_n - V_i}{R_1} + \frac{V_n - V_o}{R_2} = 0 & \quad \rightarrow \quad \frac{V_i}{R_1} + \frac{V_o}{R_2} = 0 \quad \rightarrow \quad \frac{V_o}{V_i} = -\frac{R_2}{R_1}
\end{align*}
\]

**Input and Output resistances of inverting amplifier configuration:** From the circuit,

\[
I_i = \frac{V_i - 0}{R_1} \quad \rightarrow \quad R_i = \frac{V_i}{I_i} = R_1
\]

The input impedance of the inverting amplifier circuit is \( R_1 \) (although input impedance of OpAmp is infinite).

The output impedance of the circuit is “zero” because \( V_o \) is independent of \( R_L \) (\( V_o \) does not change when \( R_L \) is changed).

**Amplifier Bandwidth:**

The voltage gain for the inverting amplifier is \(-R_2/R_1\) and is independent of frequency—same gain for a DC signal \((\omega \rightarrow 0)\) as for high frequencies. However, this voltage gain has been found using an ideal OpAmp model (ideal OpAmp parameters are independent of frequency).
A major concern for any amplifier circuit is its stability (which you will study in depth in junior and senior courses). Basically, any amplifier circuit produce a phase-shift in the output voltage. Once the phase shift becomes smaller than -180° (more negative), negative feedback becomes positive feedback. The amplifier gain should be less than 1 at these frequencies for stable operation. In a practical OpAmp, the open loop gain, $A_0$, is usually very large, ranging $10^5$ to $10^6$. To reduce the gain at high frequencies and avoid instability, the voltage gain (or voltage transfer function) of a practical OpAmp looks like a low-pass filter as shown (marked by open loop meaning no feedback). This is achieved by adding a relatively large capacitor in the OpAmp circuit chip (internally compensated OpAmps) or by providing for connection of such a capacitor outside the chip (uncompensated OpAmp). In order for the OpAmp gain to become smaller than 1 at high frequencies, the open-loop bandwidth, $f_0$ (which is the same as cut-off frequency) is usually small (10 to 100 Hz, typically).

Recall the inverting amplifier circuit discussed above. In that circuit, the voltage transfer function was independent of $A_0$ as long as $R_2/R_1 \ll A_0$. But $R_2/R_1$ is the inverting amplifier circuit gain (call it $A_1 = |V_o/V_i| = R_2/R_1$). Thus, the negative feedback worked as long as $A_1 \ll A_0$. Since $A_0$ decreases at higher frequencies, one will encountered a frequency, above which $A_1 = R_2/R1 \ll A_0$ is violated (See figure for closed-loop). Below this frequency, the amplifier gain is independent of frequency while above that frequency, we revert back to the open-loop gain curve. One can show that:

$$A_0 f_0 = A_1 f_1 = f_u = constant$$

Therefore, the product of the gain and bandwidth (which the same as the cut-off frequency) of the “amplifier circuit” is a constant and is equal to the product of open-loop gain and open-loop bandwidth. This product is given in manufacturer spec sheet for each OpAmp and sometimes is denoted as “unity gain bandwidth,” $f_u$. Note that gain in the expression above is NOT in dB, rather it is the value of $|V_o/V_i|$.

**How to solve OpAmp circuits:**

1) Replace the OpAmp with its circuit model.
2) Check for negative feedback, if so, write down $V_p \approx V_n$.
3) Solve. Best method is usually node-voltage method. You can solve simple circuits with KVL and KCLs. Do not use mesh-current method.
Physical Limitations of OpAmps

OpAmps, like other voltage amplifiers, behave linearly only under certain conditions (see page 28-29). Several limitations of OpAmp circuits are discussed before.

1. **Voltage-supply limit or Saturation:** \( v_s^- < v_o < v_s^+ \) which limits the maximum output voltage (or a for a given gain, limits the maximum input voltage). If an amplifier is saturated, one can recover the linear regime by reducing the input amplitude.

2. **Frequency Response limit:** A practical amplifier has a finite bandwidth. For OpAmps circuits with negative feedback, one can trade gain with bandwidth:

\[
A_0\omega_0 = A\omega \quad \text{or} \quad A_0f_0 = Af = f_u
\]

Note that gain in the expression above is NOT in dB, rather it is the value of \(|V_o/V_i|\).

3. **Maximum output current limit:** A voltage amplifier model (OpAmp also) includes a controlled voltage source in its output circuit. This means that for a given input signal, this controlled source will have a fixed voltage independent of the current drawn from it. If we attach a load to the circuit and start reducing the load resistance, the output voltage remains a constant and load current will increase. Following this model, one could reduce the load resistance to a very small value and draw a very large current from the amplifier.

In reality, this does not happen. Each voltage amplifier has a limited capability in providing output current. This maximum output current limit is also called the “Short-Circuit Output Current,” \( I_{SC} \). If one tries to exceed this current limit, the voltage amplifier will not behave linearly; the output current will not increase, rather the current stays constant and the output voltage and the gain will decrease.

If the a fixed load resistance, \( R_L \), is connected to the amplifier, the maximum output current means that the output voltage cannot exceed the \( R_LI_{SC} \):

\[
-R_LI_{SC} \leq v_o \leq R_LI_{SC}
\]

In this case, the maximum output current limit manifests itself in a form similar to amplifier saturation. The output voltage waveform will be clipped at value of \( R_LI_{SC} \). Two options are available to avoid the maximum current limit: (1) Increasing \( R_L \) as is seen from the above equation, (2) decreasing \( v_o \) by decreasing the input amplitude.

**Note:** The maximum output current of the OpAmp amplifier configuration is NOT the same as the maximum output current of the OpAmp itself. For example in the inverting amplifier configuration, the maximum output current of the amplifier is smaller than \( I_{SC} \) of OpAmp because OpAmp has to supply current to both the load (maximum output current of amplifier configuration itself) and the feedback resistor \( R_2 \).
4. Slew rate (Rise Time):

If input changes suddenly, the OpAmp cannot change its output instantaneously. The maximum rate of change of the output of an OpAmp is called the “slew rate” (given usually in the units of V/μs):

\[ S_0 \equiv \frac{dv_o}{dt}_{Max} \]

Slew rate affects all signals (not limited to square waves). For example, at high enough frequency and/or at high enough amplitude, a sinusoidal input turns into a triangular output signal. As an example, consider an inverting amplifier with a gain of \( A = 10 \), build with an OpAmp with a slew rate of \( S_0 = 1 \) V/μS. The input is a sinusoidal wave with an amplitude of \( V_i = 1 \) V and frequency of \( \omega \).

\[ v_i = V_i \cos(\omega t) \rightarrow v_o = -AV_i \cos(\omega t) \]

\[ \frac{dv_o}{dt} = +AV_i \omega \sin(\omega t) \rightarrow \frac{dv_o}{dt}_{Max} = AV_i \omega \]

The slew rate limit means that

\[ \frac{dv_o}{dt}_{Max} = AV_i \omega \leq S_0 \]

For the example above, \( V_i = 1 \) V, \( A = 10 \), and \( S_0 = 1 \) V/μS, we have

\[ \frac{dv_o}{dt}_{Max} = 10 \omega \leq 10^6 \rightarrow \omega \leq 10^5 \]

Which means that at frequencies above \( 10^5 \) rad/s, the output will depart from a sinusoidal signal due to the slew rate limit. Because for the sinusoidal wave, the slew rate limit is in the form \( AV_i \omega \leq S_0 \), one can avoid this nonlinear behavior by either decreasing the frequency, or by lowering the amplifier gain, or reducing the input signal amplitude.

5. Other limits: OpAmps have other physical limitations such as Input offset voltage, Input bias current, and Common-mode reject ratio (CMMR). Consult Rizzoni for a description of these limitations.
Non-inverting Amplifier

\[ V_p = V_i \]

Negative Feedback \[ \rightarrow \] \[ V_n \approx V_p = V_i \]

\[ \frac{V_n}{R_1} + \frac{V_n - V_o}{R_2} = 0 \]

Note that you should not write a node equation at OpAmp output as its a node attached to a voltage source. The value of \( V_o \) is \( AV_d \) and is indefinite. Instead of using this equation, we use \( V_d \approx 0 \) as was discussed in page 34.

Substituting for \( V_n = V_i \), we get

\[ \frac{R_2}{R_1} V_i + V_i - V_o = 0 \]

\[ \frac{V_o}{V_i} = 1 + \frac{R_2}{R_1} \]

**Input Resistance:**

\( I_i = I_p = 0 \). Therefore, \( R_i \rightarrow \infty \).

**Output Resistance:**

\( V_o \) is independent of \( R_L \), so \( R_o = 0 \).

Note that \( R_i \rightarrow \infty \) and \( R_o = 0 \) should be taken in the context that we are using an “ideal” OpAmp model. In reality, the above circuit will have input and output resistances equal to that of the OpAmp itself.

**Voltage Follower**

In some cases, we have two-terminal networks which do not match well, i.e, the input impedance of the later stage is not very large, or the output impedance of preceding stage is not low enough. A “buffer” circuit is usually used in between these two circuits to solve the matching problem. These “buffer” circuit typically have a gain of 1 but have a very large input impedance and a very small output impedance. Because their gains are 1, they are also called “voltage followers.”
The non-inverting amplifier above has $R_i \rightarrow \infty$ and $R_o = 0$ and, therefore, can be turned into a voltage follower (buffer) by adjusting $R_1$ and $R_2$ such that the gain is 1.

$$\frac{V_o}{V_i} = 1 + \frac{R_2}{R_1} = 1 \quad \rightarrow \quad R_2 = 0$$

So by setting $R_2 = 0$, we have $V_o = V_i$ or a gain of unity. We note that this expression is valid for any value of $R_1$. As we want to minimize the number of components in a circuit as a rule (cheaper circuits!) we set $R_1 = \infty$ (open circuit) and remove $R_1$ from the circuit.

**Inverting Summer**

$$V_p = 0$$

Negative Feedback: \( V_n \approx V_p = 0 \)

$$\frac{V_p - V_1}{R_1} + \frac{V_p - V_2}{R_2} + \frac{V_n - V_o}{R_f} = 0$$

$$V_o = -\frac{R_f}{R_1}V_1 - \frac{R_f}{R_2}V_2$$

So, this circuit adds (sums) two signals. An example of the use of this circuit is to add a DC offset to a sinusoidal signal.

**Non-Inverting Summer**

Negative Feedback: \( V_n \approx V_p \)

$$\frac{V_p - V_1}{R_1} + \frac{V_p - V_2}{R_2} = 0 \quad \rightarrow$$

$$V_p \left( \frac{1}{R_1} + \frac{1}{R_2} \right) = \frac{V_1}{R_1} + \frac{V_2}{R_2}$$

$$\frac{V_n - 0}{R_s} + \frac{V_n - V_o}{R_f} = 0 \quad \rightarrow$$

$$V_o = \left( 1 + \frac{R_f}{R_s} \right) V_n$$
Substituting for $V_n$ in the second equation from the first (noting $V_p = V_n$):

$$V_o = \frac{1 + R_f/R_s}{1/R_1 + 1/R_2} \left( \frac{V_1}{R_1} + \frac{V_2}{R_2} \right)$$

So, this circuit also signal adds (sums) two signals. It does not, however, inverts the signals.

**Difference Amplifier**

Negative Feedback: \( V_n \approx V_p \)

\[
\frac{V_p - V_2}{R_2} + \frac{V_p - 0}{R_3} = 0 \quad \rightarrow \quad V_n \approx V_p = \frac{R_3}{R_2 + R_3} V_2
\]

\[
\frac{V_n - V_1}{R_1} + \frac{V_n - V_o}{R_f} = 0
\]

Substituting for $V_n$ in the 2nd equation, one can get:

$$V_o = -\frac{R_f}{R_1}V_1 + \left(1 + \frac{R_f}{R_1}\right) \left(\frac{R_3}{R_2 + R_3}\right) V_2$$

If one choose the resistors such that \( \frac{R_3}{R_2} = \frac{R_f}{R_1} \), then

$$V_o = \frac{R_f}{R_1}(V_2 - V_1)$$

**Current Source**

Negative Feedback: \( V_n \approx V_p = V_s \)

$$i_L = \frac{V_n}{R} = \frac{V_s}{R} = \text{constant}$$

For a fixed value of $V_s$, the current $I_L$ is independent of value of $R_L$ and output voltage $V_o$. As such, this circuit is an independent current source.

The value of the current can be adjusted by changing $V_s$ (for a fixed $R_L$). Therefore, this circuit is also a “voltage to current” converter or a “transconductance” amplifier.
Grounded Current Source

The problem with the above current source is that the load is not grounded. This may not be desirable in some cases. This circuit here is also a current source with a grounded load if \( R_f/R_1 = R_3/R_2 \).

**Question:** Compute \( I_L \) and show that it is independent of \( R_L \).

---

**Active Filters, Integrators & Differentiators**

Consider the circuit shown. This is an inverting amplifier with impedances instead of resistors. Following the inverting amplifier solution, we find:

\[
H(j\omega) = \frac{V_o}{V_i} = -\frac{Z_2}{Z_1}
\]

Various filter circuits can be made with different choices for \( Z_1 \) and \( Z_2 \):

**1st Order Low-Pass Filter:**

\[
Z_1 = R_1
\]

\[
Z_2 = R_2 \parallel C_2 = \frac{R_2}{1 + j\omega C_2 R_2}
\]

\[
H(j\omega) = -\frac{Z_2}{Z_1} = -\frac{R_2/R_1}{1 + j\omega C_2 R_2}
\]

Compare the above voltage transfer function with the general expression for a 1st order low-pass filter:

\[
H(j\omega) = \frac{K}{1 + j\omega/\omega_c}
\]
We find that the above circuit is a low pass filter with

\[ K = -\frac{R_2}{R_1} \quad \text{and} \quad \omega_c = \frac{1}{R_2C_2} \]

The minus sign in front of \( K \) indicates an additional \(-180^\circ\) phase shift. A low-pass RC or RL filter has a phase shift of \(0^\circ\) at low frequencies and \(-90^\circ\) at high frequencies. The above amplifier has a phase shift of \(-180^\circ\) at low frequencies and \(-270^\circ\) at high frequencies (or alternatively \(+180^\circ\) at low frequencies and \(+90^\circ\) at high frequencies as we can add \(360^\circ\) to the phase angle). Another difference with passive RC or RL filters is that the gain, \( K = \frac{R_2}{R_1} \) can be set to be larger than one (i.e., amplify the signals in the pass band). As such this kind of filters are called “active filters.”

**Input Resistance:** \( R_i = R_1 \).

**Output Resistance:** \( R_o = 0 \) (OpAmp output resistance).

1st Order High-Pass Filter:

\[
Z_1 = R_1 + \frac{1}{j\omega C_1} = R_1 \left( 1 - j\frac{1}{\omega C_1 R_1} \right)
\]

\[
Z_2 = R_2
\]

\[
H(j\omega) = -\frac{Z_2}{Z_1} = -\frac{R_2/R_1}{1 - j\frac{1}{\omega C_1 R_1}}
\]

Comparing the above voltage transfer function with the general expression for a 1st order high-pass filter,

\[
H(j\omega) = \frac{K}{1 - j\omega_c/\omega}
\]

We find that the above circuit is a high-pass filter with

\[ K = -\frac{R_2}{R_1} \quad \text{and} \quad \omega_c = \frac{1}{R_1C_1} \]

Again, the minus sign in \( K \) indicates an additional \(-180^\circ\) phase shift. A high-pass RC or RL filter has a \(90^\circ\) at low frequencies and \(0^\circ\) at high frequencies. The above amplifier has a phase shift of \(-90^\circ\) at low frequencies and \(-180^\circ\) at high frequencies.

**Input Resistance:** \( R_i = Z_1 \) and \( R_i|_{\text{min}} = R_1 \).

**Output Resistance:** \( R_o = 0 \) (OpAmp output resistance).
2nd Order Band-Pass Filter:

\[ Z_1 = R_1 + \frac{1}{j\omega C_1} = R_1 \left(1 - j\frac{1}{\omega C_1 R_1}\right) \]

\[ Z_2 = \frac{R_2}{1 + j\omega C_2 R_2} \]

Defining \( \omega_c_1 = \frac{1}{R_1 C_1} \), \( \omega_c_2 = \frac{1}{R_2 C_2} \)

\[ H(j\omega) = \frac{Z_2}{Z_1} = -\frac{R_2}{R_1} \times \frac{1}{1 - j\omega/\omega_c_1} \times \frac{1}{1 + j\omega/\omega_c_2} \]

As can be seen, the voltage transfer function looks like a high-pass and a low-pass filter put together (a wide-band, band-pass filter).

To find the cut-off frequencies, bandwidth, etc. of this filter, it is simplest to write \( H(j\omega) \) in the form similar to the general form for 2nd order band-pass filters:

\[ H(j\omega) = \frac{K}{1 + jQ \left(\frac{\omega}{\omega_0} - \frac{\omega}{\omega}\right)} \]

To do so, we rearrange the terms in the expression for \( H(j\omega) \) of the above filter to get: (Trick is to make the denominator of \( H(j\omega) \) to look like \( 1 + j\ldots \))

\[ H(j\omega) = -\frac{R_2}{R_1} \times \frac{1}{1 + \frac{\omega_c_1}{\omega_c_2} + j \left(\frac{\omega}{\omega_c_2} - \frac{\omega_c_1}{\omega_c_2}\right)} = -\frac{R_2/R_1}{1 + \omega_c_1/\omega_c_2} \left(\frac{\omega_c_2}{\omega} - \frac{\omega_c_1}{\omega}\right) \]

We compare the above expression with the general form for 2nd order band-pass filters. The two equations should be identical for all value of \( \omega \). Thus:

\[ K = -\frac{R_2/R_1}{1 + \omega_c_1/\omega_c_2} \]

\[ \frac{Q}{\omega_0} = \left(1 + \frac{\omega_c_1}{\omega_c_2}\right)^{-1} \times \frac{1}{\omega_c_2} \quad \text{and} \quad Q\omega_0 = \left(1 + \frac{\omega_c_1}{\omega_c_2}\right)^{-1} \times \omega_c_1 \]
The last two equations should be solved to find \( Q \) and \( \omega_0 \):

\[
\omega_0 = \sqrt{\omega_{c1} \omega_{c2}} \quad \text{and} \quad Q = \frac{\omega_0}{\omega_{c2} - \omega_{c1}}
\]

Values of band-width, \( B \), and upper and lower cut-off frequencies can then be calculated from \( \omega_0 \) and \( Q \) values above (see pages 20 and 21).

For a wide-band band-pass filter, expressions for \( K \), \( Q \), and \( \omega_0 \) simplify considerably:

\[
\frac{\omega_{c1}}{\omega_{c2}} \ll 1 \quad \rightarrow \quad K \approx -\frac{R_2}{R_1}, \quad Q \approx \sqrt{\frac{\omega_{c1}}{\omega_{c2}}}, \quad \omega_u \approx \omega_{c2} = \frac{1}{R_2 C_2}, \quad \omega_l \approx \omega_{c1} = \frac{1}{R_1 C_1}
\]

Note that – sign in \( K \) signifies an additional -180° phase shift.

**Question:** Computer \( R_i \) and \( R_o \) of the above filter.

The above procedure can be used for any order filter. For 2nd order, band-pass filters, an alternative method exists that sometimes leads to less algebra. Consider the general form of 2nd-order band-pass filters:

\[
H(j\omega) = \frac{K}{1 + jQ \left( \frac{\omega}{\omega_0} - \frac{\omega_0}{\omega} \right)}
\]

The following observation can be made: 1) \( H(j\omega = \omega_0) \) is purely real (the imaginary part in the denominator is exactly zero). This can be used to find \( \omega_0 \), 2) \( H(j\omega = \omega_0) = K \). This can be used to find \( K \). 3) At very low frequencies, \( H(j\omega \to 0) \approx -jK\omega/(Q\omega_0) \). This can be used to find \( Q \). The first two steps are shown below.

Using the expression for \( H(j\omega) \) of our 2nd-order active bandpass-filter from previous page, we note that \( H(j\omega) \) is purely real when

\[
\frac{\omega}{\omega_{c2}} = \frac{\omega_{c1}}{\omega} \quad \rightarrow \quad \omega = \omega_0 = \sqrt{\omega_{c1}\omega_{c2}}
\]

Next, calculating \( H(j\omega = \omega_0) \), we get:

\[
H(j\omega = \omega_0) = K = -\frac{R_2}{R_1} \frac{1}{1 + \omega_{c1}/\omega_{c2}}
\]

**Important Note:** In the analysis of active filters using OpAmps, we have assumed an ideal OpAmp with an infinite bandwidth. In principle, one should include the fact that OpAmp gain will drop at high frequencies. Alternatively, one can assume an ideal OpAmp in the analysis if the bandwidth of the OpAmp is at least ten times larger than the highest frequency of interest. The bandwidth of the OpAmp should be calculated based on the gain of the circuit, \( K \)
Integrator

In the integrator and differentiator circuits below, the input is assumed to be an arbitrary function of time (it does not have to be a sinusoidal function). Voltages and currents are written in lower case to remind you of this fact.

Negative feedback  \( \rightarrow v_p = v_n = 0 \)

\[ i = C_2 \frac{dv_c}{dt} = C_2 \frac{d(v_o - v_n)}{dt} = C_2 \frac{dv_o}{dt} \]

Also,  \( i = \frac{v_n - v_i}{R_1} = -\frac{v_i}{R_1} \)

Thus:  \( -\frac{v_i}{R_1} = C_2 \frac{dv_o}{dt} \rightarrow \frac{dv_o}{dt} = -\frac{1}{R_1 C_2} v_i \)

\[ v_o(t) - v_o(0) = -\frac{1}{R_1 C_2} \int_0^t v_i(t')dt' \]

So the output of this circuit is proportional to the integral of the input.

Examples of use of such circuit include making a triangular wave from a square wave (as is done in the function generator), charge amplifiers, and analog computers (see Rizzoni for these circuits).

The problem with this circuit is that it is too good! It integrates everything including noise. Low frequency noise is a considerable problem because even small DC inputs are integrated rapidly, increasing \( v_o \), and saturating the OpAmp. Solution is to add a resistor \( R_2 \) parallel to \( C_2 \) that discharges the capacitor in long times, getting rid of integrated DC noise. The value of the resistor is chosen such that the time constant of this RC circuit \( \tau = R_2 C_2 \) is about 100 times the period of the lowest frequency signal of interest. Addition of \( R_2 \) makes the circuit look like a low-pass filter.
Differentiator

Negative feedback \[\rightarrow v_p = v_n = 0\]

\[i = C_1 \frac{dv_c}{dt} = C_1 \frac{d(v_i - v_n)}{dt} = C_1 \frac{dv_i}{dt}\]

Also, \[i = \frac{v_n - v_o}{R_2} = -\frac{v_o}{R_2}\]

Thus: \(v_o(t) = -R_2C_1 \frac{dv_i}{dt}\)

So the output of this circuit is proportional to the derivative of the input.

In practice, this circuit does not work as advertised. The problem can be seen by assuming that the input is a sinusoidal wave and finding the transfer function for the circuit:

\[H(j\omega) = \frac{V_o}{V_i} = -\frac{R_2}{1/(j\omega C_1)} = -j\omega R_2 C_1\]

As can be seen when \(\omega\) becomes large, \(V_o\) becomes large (because of derivative of \(\cos \omega t = -\omega \sin(\omega t)\)). So, practically, this circuit is not a differentiator, rather it is a “high-frequency-noise amplifier.” Bode plots of \(H(j\omega)\) shows that the transfer function is a line with a slope of +6 dB/octave.

The solution is to attenuate the amplitude of high-frequency signals by adding a resistance \(R_1\) in series with \(C_1\). At low frequencies, \(C_1\) dominate and the circuit is a differentiator. At high frequencies, \(R_1\) dominates and the circuit becomes a simple amplifier.

**Stability of OpAmp circuits with negative feedback**

As noted before, the OpAmp gain has a negative phase shift at high frequencies. If the closed-loop gain is greater than 1 when OpAmp phase shift becomes smaller than \(-180^\circ\) (becomes more negative), negative feedback becomes positive feedback and the circuit becomes unstable. Stability of feedback amplifiers will be discussed next year in the breadth courses (such as ECE102). Both active high-pass filter and differentiator circuits above are specially susceptible to this problem because of the raise in the gain at low frequencies. If a circuit encounters this problem (specially for uncompensated OpAmps), the solution is to add a capacitor \(C_2\) to the circuit as shown to make the circuit look like a band-pass filter attenuating the high-frequency signals. For the differentiator circuit above, value of \(C_2\) is chosen such that the circuit gain is less than 1 when phase shift is \(180^\circ\).