Structure of the plasma sheath in collisional to weakly collisional plasmas

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Recently, a set of two fluid equations has been derived that describes the transport of plasmas with anisotropic pressure along the lines of force and is valid for collisional to weakly collisional regimes [Phys. Fluids 29, 463 (1986); 31, 3280 (1988)]. These transport equations coupled with Poisson’s equation are used to study the structure of the nonquasineutral transition region between the plasma and the material wall. This transition region is defined as the electric sheath region. The Bohm sheath criteria has been derived by various authors by examining behavior of the solution to the plasma-sheath equation. These investigations have shown that only for a supersonic flow would an increasing electric potential (in magnitude) exist in the sheath region to accelerate the ions further toward the wall. For subsonic flows, the behavior of the electric potential is purely oscillatory, which is not physical (electrons would trap in these potential wells and wipe them out). This work shows that when a more accurate set of equations is used, the solution is purely oscillatory when the flow speed is below the ion “thermal speed.” For flow velocities in the range between the thermal and sound speed, there exist solutions where the electric field is positive definite and oscillatory but the potential is monotonic. Therefore solutions to the sheath problem exist for supersonic flow and Bohm’s sheath criteria can be violated.

I. INTRODUCTION

The macroscopic properties of plasma experiments are generally controlled and determined by the boundary conditions at the material wall. The problem of calculating the characteristics of plasma flowing along the lines of force to a material wall arises in most plasma configurations. Examples include plasma sheaths in gas discharge plasma, Langmuir probes, plasma etching devices, and the flow at the edge of fusion systems to a material wall such as limiters or divertor–collector plates. In these various contexts, plasma parameters depend on the collisional processes and structure of the electric sheath region. Here, we define the plasma sheath as the nonquasineutral transition domain bridging the main plasma region (or quasineutral region) with the physical surface. The sheath region, defined in this manner, would encompass the Debye sheath region which is usually a few Debye lengths across and where the electric potential exponentially increases.

In order to calculate the particle flow and the electric field in the sheath region, one has to solve simultaneously Poisson’s equation coupled with the full Boltzmann transport equation for the ions and electrons. The electric sheath properties in the highly collisional regime have been worked out by a number of authors. These calculations were confined exclusively to solutions of the ideal hydrodynamic (one-fluid) equation which is valid only for $\lambda / L \to 0$, where $\lambda$ is the ion mean free path and $L$ is the smallest scale length of the electric or magnetic field and the macroscopic plasma parameters. In the sheath region, $L$ is the same order as the Debye length and the ideal hydrodynamic treatment is valid only for $\lambda / \lambda_D \to 0$. It is rare that one finds a plasma that can satisfy the condition $\lambda / \lambda_D \to 0$, especially near the physical wall. Thus the application of the ideal hydrodynamic equations to the sheath problem is expected to be inaccurate.

The collisionless electric sheath has been studied by many authors using a kinetic approach. Most of the work is a generalization of the original work of Tonks and Langmuir. Tonks and Langmuir defined an integrodiifferential form (which is known as the complete plasma-sheath equation) to describe the potential distribution throughout the plasma-sheath regions, namely

$$\frac{d^2 \phi}{ds^2} + 4\pi e \int_0^s ds' \int_{-\infty}^{\infty} \frac{S_i}{v_{\|}} d^3 v - 4\pi e n_e \exp\left(\frac{e \phi}{k T_e}\right) = 0,$$

where $s$ is the arclength along the line of force, $S_i$ is the number of ions generated per second per unit volume at $s$ ($f = \int_0^s S / v_{\|} ds'$, where $f$ is the particle distribution function), and $v_{\|}$ is the velocity of the particle along the line of force. The exponential term represents the electron density, which is derived by assuming a Boltzmann distribution for the electrons, where $n_e$ is the electron density at $s = 0$ and $T_e$ is the electron temperature (assumed independent of $s$).

In order to solve the plasma-sheath equation one still has to solve for the ion source function $S_i$ through the use of additional kinetic equations. A much simpler, but less accurate, approach has been adopted in previous papers, namely, the source functions are specified rather than solved. Thus one should expect the validity and accuracy of their solution to be directly related to the choice of the plasma source function.

The previous work using kinetic theory has been limited to collisionless plasmas and the solutions are strongly dependent on the assumed source function. On the other hand,
the fluid approach (which is much simpler) has been limited to highly collisional plasmas and cannot be applied to the sheath region. In recent papers (hereafter referred to as Papers I\textsuperscript{19} and II\textsuperscript{20} respectively), we derived a new set of generalized two-fluid equations that describes plasma transport along the field lines of a space- and time-dependent magnetic field. These equations are valid for collisional to weakly collisional plasmas. They were shown to reduce to the conventional fluid equations of Braginskii\textsuperscript{21} for highly collisional plasmas and they resembled the double adiabatic or Chew, Goldeberger, and Low\textsuperscript{22} (CGL) equations in the collisionless limit with negligible heat conduction. An important feature of these equations is that the anisotropy in the ion distribution function is explicitly included. In fact, the explicit inclusion of anisotropy in the ion distribution function permits the new transport equations to be applied to a collisionless plasma. Note that standard fluid equations in the sheath region are extremely inaccurate since the electric field acts only on one degree of freedom of the distribution function and there are no forces to redistribute this change in parallel energy into the perpendicular direction. Our equations, by tracking the perpendicular and the parallel energies separately, remove this concern. Also note that in the sheath region the plasma is flowing to the wall at a very high speed, and, therefore, even for a drifting bi-Maxwellian, the number of particles with velocity in the opposite direction to the wall would be very small.

The Bohm sheath criteria has been derived by various authors\textsuperscript{1-12} by examining behavior of the solution to the plasma-sheath equation.\textsuperscript{8-18} These investigations have shown that only for a supersonic flow would an increasing electric potential (in magnitude) exist in the sheath region to accelerate the ions further toward the wall. For subsonic flows, the behavior of the electric potential is purely oscillatory, which is not physical (electrons would trap in these potential wells and remove them). In applying the new transport equations coupled with Poisson’s equation to the sheath problem, we have found that the solution is purely oscillatory when the flow speed is below the ion “thermal speed.” For flow velocities in the range between the “thermal” and sound speed, there exist solutions where the electric field is positive definite and oscillatory but the potential is monotonic. Therefore, solutions to the sheath problem exist for subsonic flow and Bohm’s sheath criteria can be violated. Such a solution, where the electric field is positive definite and oscillatory but the potential is monotonic in the subsonic region of the sheath, can only exist if one solves Poisson’s equation; assuming strict quasineutrality would not yield these results and would preclude this class of solutions. Note that violation of Bohm’s sheath criteria in other circumstances is also pointed out by Chodorak\textsuperscript{9} for cases when the magnetic field lines are not perpendicular to the wall.

In this paper the plasma source is assumed to be outside the sheath region (source-free sheath region). This assumption is different than the one used in previous papers,\textsuperscript{8-18} namely, that the source is distributed throughout the plasma.

In Sec. II, we review the transport equations derived in Paper I. The stability of the plasma-sheath transport equations is discussed in Sec. III. The numerical results are discussed in Sec. IV. The paper concludes with a summary and conclusions in Sec. V.

II. GENERALIZED TRANSPORT EQUATIONS

For completeness, we present here a brief description of the generalized transport equations derived in Papers I and II (for a detailed derivation one is referred to Papers I and II). These moment equations are derived from an underlying kinetic equation in magnetic flux coordinates expanded in the inverse of the gyrofrequency (i.e., $\Omega^{-1}$) and the Larmor radius $\rho$, retaining only the lowest-order terms. Certain approximations and assumptions are introduced in Papers I and II to close the moment equations. The electron distribution function is assumed to be isotropic, while no restriction on the ion distribution function anisotropy is imposed. In Paper I, the ion moment equations are closed by ignoring the ion conduction, which is, in general, much smaller than ion convection. In Paper II, a new electron energy flux (conduction) equation is derived by assuming a bi-Maxwellian distribution function in evaluating the fourth moments.

Generally, two distinct sets of assumptions are used in deriving the standard fluid equations.\textsuperscript{21} The first set of assumptions deals with which moments of the distribution function are needed to approximate the plasma behavior. The standard fluid equations are valid only for systems where a strong force such as collisions would make the plasma distribution isotropic (isotropic pressure and temperature). For a magnetized plasma, the magnetic field (and particle gyration around the magnetic field lines) would make that distribution isotropic in the direction perpendicular to the field line even for a collisionless plasma. Therefore, inclusion of anisotropy in particle distribution in the direction parallel to the field line would extend the validity of the fluid equations to even collisionless systems. Indeed, in Ref. 19 the new fluid equations are shown to reduce to the CGL equations in the collisionless regime. For simplified, one-dimensional sheath problems, the electric field is also acting only on one degree of freedom of the distribution function and again one can assume that the distribution function would be isotropic in the direction perpendicular to the line of force.

The second set of approximations used in the fluid equations deals with calculation of the transport coefficients, such as thermal conductivity. The form of the transport coefficients depends weakly on the exact form of the plasma distribution function. The transport coefficients derived in Papers I\textsuperscript{19} and II\textsuperscript{20} are valid over a wide range of collisionality as long as the distribution function does not deviate largely from a two-temperature drifting Maxwellian (which can be quite a deformed distribution function). Attempts were also made to ensure that in the limit of collisionless systems, the transport coefficients would not become unphysical.

In this paper, a constant magnetic field and a negligible perpendicular electric field ($E_\perp \gg E_\parallel$, where $E_\parallel$ and $E_\perp$ are, respectively, the parallel and perpendicular components of the electric field with respect to the magnetic field lines) are
assumed. In these limits, all drifts perpendicular to the lines of force vanish. Thus the problem is reduced to a one-dimen-
sional problem [i.e., flow along the lines of forces (s)], which is consistent with the assumptions made in deriving the transport equations in Papers I and II. Within these approxima-
tions, the new transport equations are valid in the sheath region and their validity is independent of the magnitude of the Debye length. These transport equations are also valid for one-dimensional sheath problems with no magnetic field such as in nonequilibrium glow discharges. In these cases the electric field acts only on one degree of freedom of the distribution function and one can assume that the distribution function would be isotropic in the direction perpendicular to the line of force.

Since almost all of the electrons entering the sheath region are reflected by the sheath potential, electrons would bounce back and forth, repeatedly, between the main plasma region and the wall (the system considered here is symmetric). Therefore, one can assume that the electron distribution function would be close to a Maxwellian. With this as-
sumption, the new generalized transport equations for a plasma with no external sources and constant magnetic field are summarized as follows: the ion continuity equation is

$$\frac{\partial n_i}{\partial t} + u_i \frac{\partial n_i}{\partial s} + n_i \frac{\partial u_i}{\partial s} = 0;$$

(2a)

the ion momentum equation is

$$m_i n_i \left( \frac{\partial u_i}{\partial t} + u_i \frac{\partial u_i}{\partial s} \right) = - \frac{\partial p_{i\parallel}}{\partial s} + q_i n_i E_{\parallel} + R_{\parallel};$$

(2b)

the ion parallel pressure equation is

$$\frac{\partial p_{i\parallel}}{\partial t} + u_i \frac{\partial p_{i\parallel}}{\partial s} = - 3 p_{i\parallel} \frac{\partial u_i}{\partial s} + \left( \frac{5}{3} p_i + \frac{2}{3} \delta p_i \right) \frac{\partial u_i}{\partial s} + \left( \frac{p_e - p_i}{\tau_{ei}} \right) + \left( \frac{p_e - p_i}{\tau_{ei}} \right);$$

(2c)

the ion perpendicular pressure equation is

$$\frac{\partial p_{i\perp}}{\partial t} + u_i \frac{\partial p_{i\perp}}{\partial s} = - p_{i\perp} \frac{\partial u_i}{\partial s} - \left( \frac{p_i - p_{i\perp}}{\tau_i} \right) \frac{\partial u_i}{\partial s} + \left( \frac{p_e - p_i}{\tau_{ei}} \right);$$

(2d)

the electron continuity equation is

$$\frac{\partial n_e}{\partial t} + u_e \frac{\partial n_e}{\partial s} + n_e \frac{\partial u_e}{\partial s} = 0;$$

(3a)

the electron momentum equation is

$$m_e n_e \left( \frac{\partial u_e}{\partial t} + u_e \frac{\partial u_e}{\partial s} \right) = - \frac{\partial p_e}{\partial s} - q_e n_e E_{\parallel} + R_e;$$

(3b)

the electron pressure equation is

$$\frac{\partial p_e}{\partial t} + u_e \frac{\partial p_e}{\partial s} = - \frac{5}{3} p_e \frac{\partial u_e}{\partial s} - \left( \frac{p_e - p_i}{\tau_{ei}} \right) \frac{\partial q_e}{\partial s};$$

(3c)

and the energy flux equation is

$$\frac{\partial q_e}{\partial t} + u_e \frac{\partial q_e}{\partial s} = q_e \left( \frac{1}{\tau_e} + \frac{8}{3} \frac{\partial u_e}{\partial s} - \frac{5}{3} \frac{p_e}{m_e} \frac{\partial T_e}{\partial s} \right).$$

(3d)

The parallel electric field $E_{\parallel}$ and the electrostatic potential $\phi$ are given by Poisson's equation:

$$\frac{\partial E_{\parallel}}{\partial s} = - \frac{\partial^2 \phi}{\partial s^2} = - 4 \pi e (n_e - n_i).$$

(4)

Here, subscripts "i" and "e" refer to ions and electrons, respectively; $n$ is the number density; $u$ is the average flow velocity along the line of force; $p_i$ and $p_i'$ are, respectively, the ion pressure parallel and perpendicular to the line of force; $p_{i\parallel}$ is the total ion pressure, $p_{i\parallel} = (p_i + p_{i\perp})/3$; and $\delta p_i$ is the anisotropy in ion pressure, $\delta p_i = (p_i' - p_i) / 3$. Also, $T_i = p_i / n_i$, $T_i' = p_{i\perp} / n_i$, and $T_{i\parallel} = p_{i\parallel} / n_i$ is the electron pressure that is assumed to be isotropic; $q_e$ is the electron conduction flux; $m_i$ and $m_e$ are the ion and electron masses; $R_{\parallel}$ is the ion-electron collisional momentum exchange rate; $\tau_{ei}$ is the electron-ion energy exchange rate; $\tau_{ri}$ is the ion pressure anisotropy relaxation time; and $\tau_{se}$ is the electron energy flux relaxation time. Expressions for $\tau_{ei}$, $\tau_{ri}$, and $\tau_{se}$ together with a detailed explanation of various terms are given in Papers I and II.

Note that the ion parallel and perpendicular pressure equations can be combined using the definitions of $p_i$ and $\delta p_i$ to produce two equations for the rate of change of the total ion pressure and ion pressure anisotropy, namely,

$$\frac{\partial p_i}{\partial t} + u_i \frac{\partial p_i}{\partial s} = - \left( \frac{5}{3} p_i + \frac{2}{3} \delta p_i \right) \frac{\partial u_i}{\partial s} + \frac{(p_e - p_i)}{\tau_{ei}}$$

(5a)

and

$$\frac{\partial \delta p_i}{\partial t} + u_i \frac{\partial \delta p_i}{\partial s} = - \left( \frac{4}{3} p_i + \frac{7}{3} \delta p_i \right) \frac{\partial u_i}{\partial s} + 3 \delta p_i \left( \frac{1}{\tau_{ei}} + \frac{1}{\tau_{ri}} \right).$$

(5b)

It is important to point out that for a quasineutral plasma ($n_i = n_e$), the ion momentum equation is singular when the ion flow speed reaches Mach 1 and the plasma flow behavior changes from subsonic to supersonic flow. It is shown in Paper I that the sound speed $C_i$ can be written as

$$C_i = \left[ (\gamma_i T_{i\parallel} + \gamma_e T_e) / (m_i + m_e) \right]^{1/2},$$

(6)

where $\gamma_e = 1$. The values of $\gamma_i$ and the sound speed as a function of the plasma collisionality are numerically calculated (Figs. 1 and 2 of Paper I). It is found that $\gamma_i$ ranges from 3 in the highly collisional regime to 3 in the collisionless limit, as expected. It will be shown in this paper that this singularity is removed if there is a finite difference between the ion and electron densities (nonquasineutral plasma).

III. STABILITY OF PLASMA-SHEATH EQUATIONS

The structure of the plasma-sheath equations derived by previous authors has shown that a certain class of solutions is not physically valid (Bohm's sheath criterion). Here, we consider a linear stability analysis of an equilibrium solution to Eqs. (2)-(4) to help us, first, identify the appropriate
boundary conditions for the sheath problem and, second, understand the plasma behavior in the sheath region. The dependent variables are separated into two parts: a uniform equilibrium part indicated by a subscript 0 and a perturbation part indicated by a subscript 1 \((\hat{A} = A_0 + \hat{A}_1 \exp[j(ks - \omega t)])\), where \(A\) represents any macroscopic quantity, and \(\hat{A} = -1\).

A dispersion relation that predicts at each point the very local behavior of the solution is derived by taking Fourier and Laplace transforms of Eqs. (2)-(4) and linearizing the results:

\[
1 = \frac{(\Omega^2 + \omega^2)(\omega^2 - k^2 C_x^2)}{(\omega^2 - \gamma^2 V_e^2)(\omega^2 - k^2 V_e^2)},
\]

where

\[
\omega^* = \omega - k u_o, \quad \lambda_D^{-2} = 4\pi n_o e^2 / T_o, \\
C_x^2 = (T_{eo} + \gamma^* T_{eo}^2) / (m_i + m_e), \quad V_e^2 = T_e / m_e, \\
V^2 = T_{eo} / m_e, \quad \gamma^* = (5 - 3\omega^*\tau_i) / (3 - j\omega^*\tau_i), \\
\Omega^2 = 4\pi n_o e^2 / m_i, \quad \omega^2 = 4\pi n_o e^2 / m_e.
\]

Note that \(\gamma^*\) defined previously \(^{19}\) is the real part of \(\gamma^*\).

To analyze the stability of the local solution to the plasma transport [Eqs. (2)-(4)] and its dependence on the flow velocity \(u_o\), we solve the dispersion relation (7) in the limit of \(\omega \to 0\):

\[
k^2 - \frac{3j(u_0^2 - 3V_e^2)}{\lambda(u_0^2 - 3V_e^2)} k^2 = \frac{(u_0^2 - C_x^2)\omega_p^2}{(u_0^2 - 3V_e^2)(u_0^2 - 3V_e^2)} k - \frac{3j(u_0^2 - C_x^2)\omega_p^2}{\lambda(u_0^2 - 3V_e^2)(u_0^2 - 3V_e^2)} = 0,
\]

where \(C_x\) and \(C_{\alpha}\) are the collisionless and collisional sound speeds, respectively,

\[
C_x^2 = (T_{eo} + 3T_{eo}^2) / (m_i + m_e)
\]

and

\[
C_{\alpha}^2 = (T_{eo} + \gamma^* T_{eo}^2) / (m_i + m_e),
\]

\[
\omega_p^2 = (\omega_p^2 + \Omega^2)\text{ is the plasma frequency, and } \lambda = u_o\tau_i \text{ is the fluid mean free path.}
\]

For weakly collisional plasma \((\lambda \to \infty)\), the roots for Eq. (8) are

\[
k = \pm \sqrt{\frac{(u_0^2 - C_x^2)\omega_p^2}{(u_0^2 - V_e^2)(u_0^2 - 3V_e^2)}} - \frac{j 2\sqrt{V_e^2(m_e/m_i)}}{\lambda(u_0^2 - C_x^2)(u_0^2 - 3V_e^2)} + O(\lambda^{-2});
\]

\[
k = j 3(u_0^2 - C_x^2) / (\lambda(u_0^2 - C_x^2)) + O(\lambda^{-2}).
\]
On the other hand, for highly collisional plasma \((\lambda \to 0)\), the roots of Eq. (8) are

\[
k = \pm \sqrt{-\frac{(u_0^2 C_2^2 \omega_p^2)}{(u_0^2 - V_e^2)(u_0^2 - 3V_e^2)}} - \frac{2\lambda V_e^2 V_e^2 (m_e/m_i) \omega_p^2}{9(u_0^2 - V_e^2)(u_0^2 - 3V_e^2)^2} + O(\lambda^2); \quad (10a)
\]

\[
k_i = \frac{3(u_0^2 - \frac{3}{2}V_e^2)\omega_p^2}{9(u_0^2 - V_e^2)(u_0^2 - 3V_e^2)^2} + O(1). \quad (10b)
\]

From Eqs. (9) and (10), we find that there are four distinct regions related to the plasma flow velocity. These regions can also be identified in Figs. 1–3, which show the solution of the dispersion relation (7) for \(\omega \to 0\) (steady state) for different collisionality regimes.

(a) Region I \((u_0^2 < \gamma, V_e^2)\): There is no physically acceptable solution for the sheath problem in the case of \(u_0^2 < \gamma, V_e^2\). This can be seen by examining Figs. 1–3 and Eqs. (9) and (10). In this region, \(k_i\) is purely imaginary and it is damped \([k_i > 0\), see Fig. 2(a)]. On the other hand, \(k^\pm\) is imaginary and approaches infinity as \(u_0^2 \to \gamma, V_e^2\). This implies that, in this region, the plasma is not quasineutral and \((n_e - n_i)\) increases exponentially in space, which is a physically unacceptable solution \((u_e \to u_i)\). Thus \(u_0^2 = \gamma, V_e^2\) represents the minimum plasma particle flow velocity.

(b) Region II \((\text{subsonic flow}, \gamma, V_e^2 < u_0^2 < C_1^2)\): For subsonic flow, \(\gamma, V_e^2 < u_0^2 < C_1^2\), \(k^\pm\) have both real and imaginary parts and they dominate the plasma behavior (see Figs. 1–3). In this region, the plasma is approximately quasineutral. The deviation from quasineutrality, \((n_e - n_i)/n_i\), is found to be of the order \(\delta T_e/\lambda\). The oscillation magnitude is found to be damped as a result of collisional effects [from Figs. 2(a) and 3(a) one can see that the imaginary part of the wavenumber \(k^\pm\) is positive]. In this region the electric field is a positive definite oscillatory function that implies that the electrostatic potential is a decreasing function with no electrostatic wells to trap any particles. As a result of the positive electric field, the ions are accelerated from their thermal velocity (subsonic) to Mach 1 speed. This region, where \(\gamma, V_e^2 < u_0^2 < C_1^2\), can be classified as the subsonic sheath region.

(c) Region III \((\text{supersonic flow}, C_1 < u_0 < V_e)\): As the flow approaches sonic speed (the Mach number approaches 1), Figs. 1–3 show that the wavelength becomes infinite \((k^\pm = 0\) and \(k_i = 0\), see Figs. 1–3) and the oscillation period approaches infinity. At this point the plasma parameters and electric field change behavior significantly \((k\) becomes imaginary with no real part). For example, the oscillatory behavior of \((n_e - n_i)\) and the electric field vanishes and the magnitude of these parameters increases significantly. This region, where \(C_1 < u_0 < V_e\), is the electrostatic Debye sheath region.

(d) Region IV \((u_0^2 > V_e^2)\): In this region, the plasma oscillations are damped in both space and time. This implies that the maximum electron flow velocity is the electron thermal velocity, \(u_0 = V_e\).

IV. NUMERICAL SOLUTIONS

To confirm the results of the stability analysis of Sec. III, we have considered a simple problem of a steady state plasma flow along the line of force to a neutralizer plate or an absorbing wall. The plasma region can be divided into two distinct regions, namely, the source region at the center of the plasma (main plasma region) and the source-free region between the source and the wall (sheath region). We have confined our analysis to the source-free region between the source and the wall, which we define as the sheath region. The new generalized transport equations, (2) and (3), coupled with Poisson's equation (4) are solved numerically as an initial value problem in space (along \(z\)). The initial conditions are the ion and electron densities, the average flow velocities, the pressures, the anisotropy in ion pressure, and the parallel electric field. At the initial point of integration we
assume that the parallel electric field is negligible \((E_\parallel^0 \approx 0)\) and the plasma is quasineutral \((n_\parallel = n_\parallel^0)\). Hereafter, subscripts \(0\) and \(w\), respectively, denote the upstream (at the sheath edge) and downstream (at the wall) parameters.

The quasineutrality condition combined with the condition of a zero net current in the plasma \((\Gamma_e = \Gamma_w = n_\parallel u_\parallel)\) results in equal ion and electron flow velocities \((u_{\parallel 0} = u_{\parallel w} = u_0)\).

To determine the maximum wall potential, at least one of the plasma parameters at the wall has to be specified. It would be appropriate to assume that the ambipolar potential developed near the sheath would only impede the electron flow velocity before the electrons reach the wall, while right at the wall electrons should flow at their thermal speed. This is also confirmed by Figs. 2 and 3 which predict no physical solution for electron speeds above the electron thermal velocity. Also note that the electron momentum equation, Eq. (3b), is singular at \(n_{\text{max}} = n_{\text{ew}} = V_e\). This result allows one to estimate both the maximum electron flow velocity and the electrostatic potential at the wall.

The numerical solutions show that the plasma flow velocity at the entrance of the sheath depends on the plasma collisionality, plasma parameters at the entrance of the sheath (such as \(T_e/T_i\)), and also on the total sheath length of the sheath region. We have found that stable numerical solutions can only be found for cases with initial plasma flow velocity \(u_{\parallel 0} \sqrt{3V_e}\) as predicted by the stability analysis of Sec. III.

Next, in order to quantify the dependence of the sheath entrance velocity on the total length of the sheath region we have considered cases with \(T_0 = T_{\parallel 0}\), with a small but finite pressure anisotropy, \(\delta p_{\parallel 0}/p_{\parallel 0} = 0.05\), and different collisionality regimes. Given the small variation in the plasma parameters (temperature and anisotropy) in the sheath region where flow is subsonic (the region outside the “Debye” sheath), one would expect that, for a given sheath length, the sheath entrance velocity and the potential drop across the sheath could be found by examining our results and finding the appropriate parameters at the given distance from the wall.

Numerical solutions of Eqs. (2)-(4) for a plasma in the weakly collisional regime are shown in Figs. 4-7. Parameters plotted are the quasineutrality \((n_i - n_e)\) (Fig. 4), electric field (Fig. 5), electrostatic potential (Fig. 6), and ion flow velocity (Fig. 7).

The deviation from quasineutrality is measured by calculating the difference between the ion and the electron densities and is plotted in Fig. 4 as a function of length along the line of force. The deviation from quasineutrality in the subsonic sheath region \((u_0 < C_i)\) is found to be small but finite. It is also found that the difference between the ion and the electron densities, \((n_i - n_e)/n_i\), is sensitive function of collisionality and ion flow velocity. For subsonic flow, \((n_i - n_e)/n_i\) is found to be an oscillatory function and the magnitude of these oscillations is found to be inversely proportional to the collisional mean free path. As the charged particles move toward the wall, the oscillation magnitude decreases due to collisional damping (see Fig. 4). On the other hand, the oscillation period increases with increasing flow velocity and, as the flow approaches sonic flow, the oscillation period approaches infinity \((k^{-1} \to 0)\). At this stage, the parameter \((n_i - n_e)/n_i\) changes behavior from oscillatory to exponential. For supersonic flow (in the sheath region), \(n_i \gg n_e\) (see Fig. 6).

The fact that the deviation from quasineutrality is small for subsonic flow is not surprising. Large deviations would produce a very large electric field. However, the phenomena discussed here can only be derived if one solves Poisson’s equation. Assuming strict quasineutrality\(^{6-18}\) would not yield these results and would preclude this class of solutions.

The finite difference between the ion and electron densities in the subsonic sheath region gives rise to a finite, positive definite, oscillatory, electric field (Fig. 5). This result is very critical because it implies that there are no electrostatic wells to trap the plasma particles.

The distinct differences in the characteristics of the parameter \((n_i - n_e)\) and the electric field in the subsonic and supersonic sheath regions also translate into significant differences in the profiles of the electrostatic potential as the particles accelerate from subsonic to supersonic speeds. In the subsonic sheath region, the average electrostatic potential (averaged over one oscillation period) is found to increase rather slowly (linearly) with space, while in the electrostatic Debye sheath region, it increases exponentially with space (Fig. 6).

The scaling of the subsonic sheath in comparison to the electrostatic Debye sheath can be found from the ratio of the electric field in the subsonic sheath region to the electric field in the electrostatic Debye sheath region. In the subsonic sheath region, for a fixed initial ion pressure anisotropy, the magnitude of the electric field is found to be inversely proportional to the ion pressure anisotropy relaxation length, \(\lambda_f = u_0/T_i\). On the other hand, the magnitude of the electric field in the electrostatic Debye sheath region is found to be inversely proportional to the Debye length \((\varepsilon E_k/T_e\).

![Graph](https://example.com/graph.png)

**FIG. 4.** Normalized plasma density variation, \((n_i - n_e)/n_i\), in the sheath region as a function of the distance along the line of force, \((z)/\lambda_0\). Ratio of the Debye length (at the sheath edge) to the initial fluid mean free path, \(\lambda_f/\lambda_0 = 0.002\). The initial ion pressure anisotropy \(\delta p_\alpha/p_\alpha\) is 0.05. The ion and electron temperatures at the sheath edge are assumed equal \((T_\alpha = T_e\)). The point \(s = 0\) corresponds to the position of the wall.


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FIG. 5. Normalized electric field \((eE_{D}/T_e)\) in the sheath region as a function of the distance along the line of force, \((s)/\lambda_D\). Ratio of the Debye length (at the sheath edge) to the initial fluid mean free path, \(\lambda_D/\lambda_{io}\), is 0.002. The initial ion pressure anisotropy, \(\delta p_i/p_{io}\), is 0.05. The ion and electron temperatures at the sheath edge are assumed equal \((T_e = T_{io})\). The point \(s = 0\) corresponds to the position of the wall.

\[\sim \lambda_{D}^{-1}\] Thus one can conclude that the ratio of the electric field in the subsonic sheath region to the electric field in the electrostatic Debye sheath region is proportional to \(\lambda_{D}/\lambda_p\), which is, in general, much smaller than 1 \((\lambda_{D}/\lambda_p < 1)\). Thus the ratio of the scaling of the subsonic sheath to the supersonic sheath should be of the order \(\lambda_p/\lambda_{D}\). Note that the scaling of the subsonic sheath region also depends on the initial ion pressure anisotropy.

The ion flow velocity plays a significant role in the analysis of the plasma-sheath problem. For example, in the previous work mentioned earlier, it is assumed that a stable sheath can only exist if the ion flow velocity at the sheath edge is at least equal to the sound speed. In this paper, we have found that one can obtain a stable solution to the sheath problem without satisfying Bohm’s sheath criterion (a stable solution exists for \(M_e < 1\), subsonic flow). The major assumption in the previous work (e.g., Bohm’s analysis) is that the plasma is considered collisionless. For a collisionless plasma and subsonic flow, we have found (see Fig. 1) that the wavenumber \(k\) is real with no imaginary part. This result implies that there is no mechanism to increase the ion flow velocity from subsonic to supersonic speed. Thus, in an experiment with a finite plasma dimension (length), and where the plasma is collisionless, the ions must flow at a speed greater than or equal to sound speed. This result is the equivalence of the Bohm sheath criterion. Note that ignoring collisional effects resulted in a set of equations that has an unstable solution for \(M_e < 1\). In Sec. III, we have found that the collisional effects (independent of how small they are, as long as they are finite) are sufficient to stabilize the solution of the sheath problem for subsonic flow. To understand why such a small term has a significant effect on the sheath problem, one has to realize that the collisional effects tend to perturb the ion and electron densities, thus initiating the generation of ion acoustic type standing waves in the plasma. The positive definite electric field generated by the ion acoustic type standing waves is responsible for accelerating the ions from their thermal speed to the speed of sound (see Figs. 2 and 3). In general, the introduction of any driving force capable of perturbing the ion and electron densities (generating ion acoustic type standing waves) will be sufficient to stabilize the solution for subsonic flow. For example, in this paper we have found that the collisional effects (ion-drag, \(\sim \lambda_D^{-1}\)) play a significant role in the stability of the sheath region.
ion, ion-electron, and ion-neutral collisions) are responsible for the generation of the ion acoustic type standing waves. Another example was given by Chodura, in which he found that the inclusion of a finite $\mathbf{v} \times \mathbf{B}$ force in the ion momentum equation is sufficient to stabilize the sheath solution for subsonic flow. In view of the above, the nonsatisfaction of Bohm's criterion should be considered more as a warning against a breakdown of the previous theoretical models (absence of driving forces) than as a condition on the system itself.

V. SUMMARY AND CONCLUSIONS

The new generalized transport equations derived in Paper I coupled with Poisson's equation are used to study the structure of the electric sheath. The analysis in this paper is restricted to the case of no distributed sources in the sheath region. This assumption is different than the one assumed in previous papers, namely, that the source is distributed throughout the plasma. The effects of collisionality on the structure of the sheath region are investigated. Numerical solutions for the flow behavior have been obtained.

The Bohm sheath criteria has been derived by various authors by examining behavior of the solution to the plasma-sheath equation. These investigations have shown that only for a supersonic flow can an increasing electric potential (in magnitude) exist in the sheath region to accelerate the ions further toward the wall. For subsonic flows, the behavior of the electric potential is purely oscillatory, which is not physical (electrons would trap in these potential wells and wipe them out). Our work shows that when a more accurate set of equations is used, the solution is purely oscillatory when the flow speed is below the ion thermal speed. For flow velocities in the range between the thermal and sound speed, there exist solutions where the electric field is positive definite and oscillatory but the potential is monotonic. Therefore, solutions to the sheath problem exist for subsonic flow and Bohm's sheath criteria is not strictly correct. Note that this violation of Bohm's sheath criteria is also pointed out by Chodura for cases when the magnetic field lines are not perpendicular to the wall.

The transition layer between a plasma and an absorbing wall is shown to be composed of a subsonic sheath, which is a sensitive function of collisional effects and flow velocity, and scales as $\lambda_D/\lambda_p$ and the electrostatic Debye sheath of several Debye lengths, in which the flow is supersonic. In the subsonic sheath region, the collisional effects (ion pressure anisotropy relaxation and ion-electron energy rethermalization processes) are found to perturb the ion and electron densities. These perturbations result in ion acoustic type standing waves. These standing waves are found to be responsible for a finite and always positive (but oscillatory) electric field that accelerates the ions from subsonic to the sound speed. An analytical expression for the wavenumber (oscillation period) and magnitude of these oscillations is given in Sec. III.

The solution changes behavior when the flow velocity changes from subsonic to supersonic and a nonoscillatory electric field accelerates the ions in the electrostatic Debye sheath region. Also, the sensitivity of the solution to collisional effects and to flow velocity will vary significantly from highly sensitive in the subsonic sheath to insensitive in the electrostatic Debye sheath region.

The equations also predict a maximum flow velocity in the sheath region which is equal to the electron thermal speed. For a flow velocity greater than the electron thermal speed, the solution is found to be damped in space and time. Similarly, in the case of no distributed sources in the sheath region, the equations predict a minimum flow velocity in the sheath region which is equal to the ion thermal speed $\frac{m_i}{T_i}$, $\frac{1}{\gamma} (\frac{m_i}{T_i})^{1/2}$. These predictions help in understanding the structure of the electric sheath region.

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