Introduction

Frequency Domain

In principle, the voltages and currents in analog circuits are arbitrary functions of time (we call them signals or waveforms). Analytical analysis of the circuit response to an arbitrary input waveform is difficult and requires solution to a set of differential equations. Even numerical analysis becomes difficult when there are a lot of circuit elements. Fortunately, there are ways to find the response of a linear circuit to time-dependent signal. These approaches are based on the following observations:

1. For circuits driven by sinusoidal sources, the forced response of the state variables (currents and voltages) are all sinusoidal functions with the same frequency as the source. We can solve these circuits easily using Phasors, for example

This is derived from the mathematical properties of sinusoidal functions. Forced response of a set of linear differential equation (circuit equations) to a sinusoidal function is a sinusoidal function. This property leads to special analysis tools for AC circuits using “phasors,” or using Fourier transform. AC steady-state analysis of linear circuits are covered in 60A/B. When we use phasors, the circuit equation do not contain time anymore, but they include frequency $\omega$. As such, this is usually called analysis in “frequency-domain” to differentiate that from “time-domain” analysis where we solve the differential equation to find the circuit response.

2. Any arbitrary but periodic signal can be written as a sum of sinusoidal functions using Fourier series expansion.

For example, a square wave with period $T$ or frequency $\omega_0 = (2\pi)/T$ and amplitude $V_m$ can be written as:

$$v(t) = \frac{4V_m}{\pi} \left[ \sin(\omega_0 t) + \frac{1}{3} \sin(3\omega_0 t) + \frac{1}{5} \sin(5\omega_0 t) + \ldots \right]$$

Signals with frequencies $n\omega_0$ ($n$ integer) are called harmonics of the fundamental frequency, $\omega_0$. In general the amplitude of higher harmonics become smaller as $n$ become larger. The idea of decomposition of a periodic function to a sum of sinusoidal functions can be extended to an arbitrary temporal function by using Fourier integrals. As such, in principle, any function of time can be written as a sum of (or an integral of) sinusoidal functions.

3. Proportionality and superposition principles state that response of a linear circuit to a linear combination of sources is equivalent to the linear combination of circuit response to each individual source.
Basically, in a circuit with several independent sources, the value of any state variable equals to the algebraic sum of the individual contributions from each independent source. So, in a circuit with a time-dependent source, we can use Fourier series decomposition and replace the source with a linear combination of several sinusoidal sources. We can then find the response of the circuit to each sinusoidal source and then use proportionality and superposition to find the response to the time-dependent source.

For example, suppose we want have a circuit driven by a source that can be decomposed into \( v(t) = A \cos(100t) + B \cos(300t) \). We want to know the voltage across an element, \( \bar{v}(t) \). We solve the circuit with the source \( \cos(100t) \) and find the voltage across the element interest, suppose \( \alpha \cos(100t + \phi_\alpha) \). We then repeat the analysis with a source \( \cos(300t) \) and find the voltage across the element interest, suppose \( \beta \cos(300t + \phi_\beta) \). The response of the circuit to \( v(t) = A \cos(100t) + B \cos(300t) \), then is \( \bar{v}(t) = A\alpha \cos(100t + \phi_\alpha) + B\beta \cos(300t + \phi_\beta) \).

The problem is actually much simpler than the example above. In principle, solution of AC steady-state circuit is simple and we typically find the response the circuit with frequency, \( \omega \) as a parameter. We can then construct the response by replacing \( \omega \) with frequencies of interest in the response equation (e.g., set \( \omega = 100 \) and 300 in the above example). Another major simplification arises when the circuit response is frequency independent. In that case, the circuit response can be directly applied to any time-dependent function. For example, in the above example, if the circuit response to \( \cos(100t) \) and \( \cos(300t) \) sources were, respectively, \( \alpha \cos(100t) \) and \( \alpha \cos(300t) \) (frequency independent), then the circuit response is simply: \( \bar{v}(t) = \alpha v(t) \).

Therefore, we focus on circuits driven by sinusoidal sources. We solve these circuits in frequency domain. We try to find circuit parameters with frequency \( \omega \) as a parameter to facilitate construction of response to an arbitrary function of time. In particular, we are very interested in regimes in which the circuit response is independent of the frequency as the output wave-forms will be identical to input signals.

There are several ways to solve the circuit in frequency domain, all having same mathematical foundation. We can use phasors (which are really Fourier Transforms). Or, we can use complex frequency domain which is sometimes called “s-domain” \((s = \sigma + j\omega)\). In junior level courses and beyond, you will probably use complex frequency domain mainly. Circuit analysis with phasors is sufficient for the work we do in this class.

Analysis in frequency domain is straight-forward. Resistors, capacitors, and inductors are replaced by impedances, \( Z \): \( Z = R \) for a resistor, \( Z = 1/(j\omega C) \) for a capacitor and \( Z = j\omega L \) for an inductor. Impedances obey Ohm’s Law: \( V = ZI \). Thus, with impedances, the circuit reduces to a “resistive” circuit and all analysis techniques of resistive circuits (node-voltage
method, mesh-current method, Thevenin Theorem, etc.) apply. The only difference is that analysis is performed using complex variables.

**Circuit Components**

It is not practical to design a complete circuit as a whole from scratch. It is usually much easier to break the circuit into components and design and analyze each component separately. In this manner we can design “building blocks” (such as amplifiers, filters, etc.) that can be used in a variety of devices. A typical analog circuit is composed of a “source,” one or several “two-port networks,” and a “load.”

![Circuit Components Diagram]

Thevenin Theorem provides the corner stone of the strategy to divide the circuit into components and analyze each independently. We can see this by examining the Thevenin Theorem.

**Thevenin Theorem and Thevenin or Norton Equivalents**

![Thevenin and Norton Diagram]

We know from linear circuit theory (See Textbook for 60A) that the $IV$ characteristics of a two-terminal network is in the form of (using active sign convention):

$$V = V_t - Z_t I; \quad Z_t = Z_n; \quad I_n Z_n = V_t$$

which is similar to the $IV$ characteristics of the Thevenin or Norton forms shown above. Therefore, any two-terminal network can be replaced by its Thevenin or Norton equivalent. An important corollary to the Thevenin Theorem is that if a two-terminal network does not include an “independent source” it will be reduced to a single “impedance” (even if it includes dependent sources).

**How to calculate the Thevenin equivalent:** You have seen a detailed discussion of Thevenin/Norton forms in ECE60A. In summary, the best method is to calculate two of the following three parameters: (1) Open-circuit voltage, $V_{oc}$ (found by setting $I = 0$), (2)
Short-circuit current, $I_{sc}$ (found by shorting the terminals of the two-terminal network, i.e., setting $V = 0$), and (3) Direct calculation of $Z_t$ which is the resistance seen at the terminal with the independent sources “killed” (i.e., their strength set equal to zero). Remember, you should NOT kill dependent sources. The usual “rule of thumb” is to find $V_{oc}$ and $I_{sc}$ if there is a dependent source in the problem, and to find $V_{oc}$ and $Z_t$ if there is no dependent source in the problem. Then, one can find the Thevenin and Norton parameters from:

$$V_t = V_{oc}; \quad I_n = I_{sc}; \quad I_nZ_t = V_t$$

**Example:** Find the Thevenin and Norton Equivalent of the circuit below:

![Image of circuit](image)

We need to find two of the three parameters $V_{oc}$, $I_{sc}$, and $Z_t$ ($R_t$ here). It is best to find $V_{oc}$ and $Z_t$ for this problem (no dependent source) but all three are calculated for demonstration of the solution technique.

1. $V_{oc}$: Using node-voltage method and noting that since $I = 0$, by KVL, $V_1 = V_{oc}$.

$$\frac{V_1 - 25}{5} - 3 + \frac{V_1}{20} = 0$$

$$4V_1 - 100 - 60 + V_1 = 0$$

$$V_1 = 32V \quad \rightarrow \quad V_{oc} = V_1 = 32V$$

2. $R_t$ (killing the independent sources)

From the circuit, we have

$$R_t = 4 + (5 \parallel 20) = 4 + 4 = 8 \Omega$$

3. $I_{sc}$ To calculate $I_{sc}$ by nodal analysis, note that $I_{sc} = V_1/4$. Then,

$$\frac{V_1 - 25}{5} + \frac{V_1}{4} - 3 + \frac{V_1}{20} = 0$$

$$4V_1 - 100 + 5V_1 - 60 + V_1 = 0$$

$$V_1 = 16V \quad \rightarrow \quad I_n = I_{sc} = \frac{V_1}{4} = 4A$$

So, the Thevenin/Norton parameters are: $V_t = 32 V$, $I_n = 4 A$, and $R_t = 8 \Omega$. (note, $V_t = I_nR_t$.)

*ECE60L Lecture Notes, Winter 2003*
How to measure the Thevenin equivalent: Suppose we have given a box with two terminals and want to find the Thevenin equivalent of the circuit inside the box. In principle, we cannot use the above technique and try to measure $V_{oc}$, $I_{sc}$, and $Z_t$. We cannot turn off the input signal and use a ohm-meter to measure $R_t$. Nor can we short the terminals and measure $I_{sc}$ (there is a good chance that we are going to ruin the circuit if we do that). In principle, we can use a volt-meter (or scope) to measure $V_{oc}$ but care should be taken as it is not known a priori if the internal resistance of the volt-meter (or scope) is large enough to act as an open circuit (there are other complications). There is also the issue of measurement error that one should consider.

Instead of measuring $V_{oc}$, $I_{sc}$, and $Z_t$ directly, it is best to measure the $IV$ characteristics of the two-terminal network. We can do this by attaching a variable load (a resistance) to the box, vary the load which changes the output voltage and currents, and measure several pair of $I$ and $V$. These data point should lie on the $IV$ line of the two-terminal network. Values of $V_t$, $I_n$, and $R_t$ can be read directly from the graph as shown. This method is much better as by using a “best-fit” line to our data, we can minimize random measurement errors.

How to find the Thevenin equivalent using PSpice: You can use the same technique described above for measuring the Thevenin parameters with PSpice. Attach a “variable” load to the circuit. Ask PSpice to compute output voltage $V$ as a function of load resistance $R_L$. Use PROBE to plot the output current $I$ versus the output voltage $V$ and you will have the $IV$ characteristics of the circuit.
How each sub-circuit sees other elements

The strategy of dividing a circuit into individual components works because of the Thevenin Theorem. Recall that any two-terminal network can be replaced by its Thevenin equivalent. In addition, if a two-terminal network does not include an “independent source” it will be reduced to a single “impedance” (even if it includes dependent sources).

What Source sees: The source sees a two-terminal network. This two-terminal network does not contain an independent source. So it can be reduced to a single impedance.

What Load sees: The load sees a two-terminal network. This two-terminal network contains an independent source. So it can be reduced to its Thevenin equivalent.

What each two-port network sees: Following the logic above, it’s obvious that each two-port network sees a two-terminal network containing an independent source in the input side (can be reduced to a Thevenin form) and a two-terminal network that does not contain an independent source on the output side (so it can be reduced to a single impedance).

The above observations indicate that we do not need to solve a complete circuit. For example, for a particular two-port network, we only need to solve the circuit above (with $V_s$, $Z_s$, and $Z_L$ as parameters). Then, wherever this two-port network appears in a circuit, we can use these results.
Two-port networks

As you can imagine, majority of components in electronic circuits are two-port networks. For example, in a tape recorder, a large number of two-port networks exists between the source (tape head) and the load (speakers). They amplify the signal, filter out the unwanted noise, and process the signal. We study several two-port networks in 60L. As we noted in the previous page, we can design and analyze these two-port networks using a simple model for the previous stages and a load impedance for later stages of the system as is shown below.

Consider the two-port network below. It “communicates” with the outside world (rest of the circuit) through 4 parameters: $V_i$, $I_i$, $V_o$, and $I_o$. If we solve the two-port network circuit once and find the relationship between these four parameters, we do not need to do that again. While any linear two-port network can be reduced to a combination of four elements (see your circuit theory textbook), it is customary to use the following parameters to describe the behavior of a two-port network.

Voltage transfer function, $H_v(j\omega) = \frac{V_o}{V_i}$

Current transfer function, $H_i(j\omega) = \frac{I_o}{I_i}$

Equivalent input impedance, $Z_i(j\omega) = \frac{V_i}{I_i}$

Equivalent output impedance, $Z_o(j\omega) = \frac{V_o}{I_o}$

The equivalent output impedance as defined above is the equivalent Thevenin impedance of a two-terminal network consisting of our 2-port network, $Z_s$, and $V_s$.

What are $Z_L$ and $Z_s$: Consider a circuit in which our two-port network above is the “nth” two-port network (see figure in the previous page). In this case, the output voltage of “n-1” two-port network is the same as the input voltage of our “nth” two port network: $V_o|_{n-1} = V_i|_n$ and the output voltage of our “nth” two-port network is the input voltage to the “(n+1)th” two-port network: $V_o|_n = V_i|_{n+1}$ (with the similar relationship between the currents). The transfer function definitions above indicate that $Z_L$ is actually the input impedance of “(n+1)th” two-port network (next stage) and $Z_s$ is the output impedance of “(n-1)th” two-port network (previous stage).

Interaction between Components & Voltage and Power Transfer

It is obvious from the definitions of transfer functions above that they depend on the values of $Z_L$ and $Z_s$. This means that when a two-port network is placed in a circuit, the output impedance of the previous stage ($Z_s = Z_i|_{n-1}$) and the input impedance of the next stage ($Z_L = Z_i|_{n+1}$) affect the two-port network transfer functions.
One is interested to find if there is a way to ensure good coupling and maximum signal transfer between connecting two-port networks. Consider the connection between two two-port network as is shown below.

As we are interested in the interaction between the two-port network, we replace the “nth” two-port network and all of the circuit to its left with its Thevenin equivalent. From definitions of transfer functions above, we know that $Z_s$ is actually the output impedance of the “nth” two-port network $Z_s = Z_{o|n}$. Similarly the “(n+1)th” two-port network and all of the circuitry to its right can be replaced by its Thevenin equivalent, $Z_L$ (as there is no independent source in there). Again, we note that $Z_L = Z_{l|n+1}$

Good coupling between components typically means largest $I_L$, $V_L$ or power, $P_L = V_L I_L$. Unfortunately, these three parameters do not maximize simultaneously.

$$I_L = \frac{V_s}{Z_s + Z_L}$$

$$V_L = \frac{Z_L}{Z_s + Z_L} V_s$$

$$P_L = V_L I_L = \frac{Z_L}{(Z_s + Z_L)^2} V_s^2$$

Values of $I_L$, $V_L$, and $P_L$ are plotted in the figure assuming $V_s$ and $Z_s$ are fixed. We can see that best current coupling (maximum $I_L$) when $Z_L = 0$ (or effectively, $Z_L/Z_s \ll 1$) and the best voltage coupling (maximum $V_L$) when $Z_L \to \infty$ (or effectively, $Z_L/Z_s \gg 1$). The best power coupling (maximum $P_L$) is somewhere in between when $Z_L = Z_s^*$ ($R_L = R_s$ and $X_L = X_s$).

Maximum voltage transfer: $\frac{Z_L}{Z_s} \gg 1 \rightarrow V_L|_{max} = V_s$

Maximum current transfer: $\frac{Z_L}{Z_s} \ll 1 \rightarrow I_L|_{max} = \frac{V_s}{Z_s}$

Maximum power transfer: $Z_L = Z_s^* \rightarrow P_L|_{max} = \frac{V_s^2}{4Z_s}$

Maximum power transfer is not usually a criteria for coupling components (except the last stage of coupling to the load). In most cases, we are interested in good voltage coupling to
keep power dissipation in the circuit small. For example, consider a CD player. The source produces a low-voltage signal proportional to the information on the CD which needs to be amplified, translated into sound frequencies, amplified further, filtered, etc. and then fed to a speaker. In order to keep the circuit small and cheap, we amplify the signal and do the signal processing with signals of substantial voltage but low current. This keep the power dissipation in each stage small. Only in the last stage (power amplifier) the signal current is increased to drive the load (speakers).

The criteria for best voltage coupling is $Z_L \gg Z_s (V_L \approx V_s)$. If we are modeling the interaction between two two-port networks, $Z_s$ represent the output impedance of the previous stage, $Z_L$ represents the input resistance of the next stage. Therefore, best voltage coupling condition translates into ensuring that output impedance of previous stage is much smaller than input impedance of the next stage.

$$Z_o|_n \ll Z_i|_{n+1}$$

And, a useful goal for designing two-port networks is to ensure that input impedance is large and output impedance is small.

In this course, we examine many two-port networks, calculate their parameters (transfer functions and input and output impedances), and experiment with them in the Lab.

We showed that in general we need to solve a circuit as is shown. Examination of the definitions of parameters of the two-port network shows that only $Z_o$ depends on $Z_s$. Furthermore, if we follow the good practice of designing circuits with low output impedance and high input impedance, one can easily show that $Z_o$ will become independent of $Z_s$. So, for the rest of this course, $Z_s$ is ignored (then, $V_s = V_i$). Furthermore, to facilitate understanding of circuit behavior, we will first solve all of the circuits assuming $Z_L \rightarrow \infty$ (this is the same as setting $Z_o|_n \ll Z_i|_{n+1}$). We will then investigate the impact of adding a load to the circuit (called terminated networks).
Voltage divider as an example of a two-port network

\[ I_i = \frac{V_s}{R_s + R_1 + R_2 \parallel R_L} \]
\[ I_o = -\frac{V_o}{R_L} \]
\[ V_o = (R_2 \parallel R_L) I_i = \frac{R_2 \parallel R_L}{R_s + R_1 + R_2 \parallel R_L} V_s \]
\[ V_i = (R_1 + R_2 \parallel R_L) I_i = \frac{R_1 + R_2 \parallel R_L}{R_s + R_1 + R_2 \parallel R_L} V_s \]

From the above we can now calculate the parameters of our two-port network:

\[ H_o = \frac{V_o}{V_i} = \frac{R_2 \parallel R_L}{R_1 + R_2 \parallel R_L} \]
\[ H_i = \frac{I_o}{I_i} = \frac{R_2 \parallel R_L}{R_L} \]
\[ Z_i = \frac{V_i}{I_i} = R_1 + R_2 \parallel R_L \]
\[ Z_o = \frac{V_o}{I_o}_{V_i=0} = (R_s + R_1) \parallel R_2 \]

Note that the transfer functions are independent of the value of input signal strength, \( V_i \) (because the circuit is linear) and the parameters of the two-port network “depend” on values of \( R_s \) and \( R_L \). For this case, the transfer functions are frequency independent (no capacitor or inductor in the circuit).

Condition of best voltage coupling is \( R_L \gg Z_o \) (this means that \( H_v \) will be independent of \( R_L \) and \( H_v \) is at its maximum value). For simplicity, let’s consider a case with \( R_s = 0 \). In this case,

\[ R_L \gg Z_o = \frac{R_1 R_2}{R_1 + R_2} \rightarrow R_L R_1 + R_L R_2 \gg R_1 R_2 \]

Replacing for \( R_2 \parallel R_L = R_2 R_L/(R_2 + R_L) \) in \( H_v \), we get

\[ H_v(j\omega) = \frac{R_2 R_L}{R_1 R_2 + R_1 R_L + R_2 R_L} \]

Using \( R_L R_1 + R_L R_2 \gg R_1 R_2 \), we can drop \( R_1 R_2 \) term in the denominator of \( H_v \):

\[ H_v(j\omega) \approx \frac{R_2 R_L}{R_1 R_L + R_2 R_L} = \frac{R_2}{R_1 + R_2} \]

which is independent of \( R_L \). Note that the value of \( H_v \) is exactly the value of \( H_v \) in absence of any load (\( i.e., R_L \rightarrow \infty \)). It is also straight forward to show that this value represent the maximum value of \( H_v \) in the presence of any load and, thus, represents the best voltage coupling.

ECE60L Lecture Notes, Winter 2003